Walks in the quadrant: differential algebraicity

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with

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Counting quadrant walks... at the séminaire lotharingien

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Counting quadrant walks

Let $S$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{N}^2$ (starting point).

Example. $S = \{10, \bar{1}0, \bar{1}\bar{1}, \bar{1}1\}$, $p_0 = (0, 0)$
Counting quadrant walks

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- What is the number $q(n)$ of $n$-step walks starting at $p_0$ and contained in $\mathbb{N}^2$?
- For $(i, j) \in \mathbb{N}^2$, what is the number $q(i, j; n)$ of such walks that end at $(i, j)$?

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The associated generating function:

$$Q(x, y; t) = \sum_{n \geq 0} \sum_{i, j \geq 0} q(i, j; n)x^i y^j t^n.$$ 

What is the nature of this series?
A hierarchy of formal power series

- **Rational series**
  \[ A(t) = \frac{P(t)}{Q(t)} \]

- **Algebraic series**
  \[ P(t, A(t)) = 0 \]

- **Differentially finite series (D-finite)**
  \[ \sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0 \]

- **D-algebraic series**
  \[ P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \]

Multi-variate series: one DE per variable
1. Write a functional equation

Example: \( S = \{01, \bar{1}0, 1\bar{1}\} \)

\[
Q(x, y; t) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)
\]

with \( \bar{x} = 1/x \) and \( \bar{y} = 1/y \).

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or

\[
(1 - t(y + \bar{x} + x\bar{y})) Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),
\]

The polynomial \( 1 - t(y + \bar{x} + x\bar{y}) \) is the kernel of this equation.

The equation is linear, with two catalytic variables \( x \) and \( y \) (tautological at \( x = 0 \) or \( y = 0 \)) [Zeilberger 00].
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or
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(1 - t(y + \bar{x} + x\bar{y})) xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)
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- The equation is linear, with two catalytic variables \( x \) and \( y \) (tautological at \( x = 0 \) or \( y = 0 \)) [Zeilberger 00]
Equations with **one** catalytic variable are much easier!

**Theorem [mbm-Jehanne 06]**

Let $P(t, y, S(y; t), A_1(t), \ldots, A_k(t))$ be a polynomial equation in one catalytic variable $y$ that defines uniquely $S(y; t), A_1(t), \ldots, A_k(t)$ as formal power series. Then each of this series is algebraic.

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The proof is constructive.

**Example:** for \( S(y; t) = Q(0, y; t) \),

\[
\frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyS(y; t) + \frac{1}{y} \right)^2 - \left( tyS(y; t) + \frac{1}{y} \right) - 2t^2 S(0; t).
\]
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The proof is constructive.

⇒ A special case of an Artin approximation theorem with “nested” conditions [Popescu 86]
Equations with two catalytic variables are harder...

D-finite transcendental

\[(1 - t(y + \bar{x} + x\bar{y})) xyA(x, y) = xy - tyA(0, y) - tx^2A(x, 0)\]

Algebraic

\[(1 - t(\bar{x} + \bar{y} + xy)) xyA(x, y) = xy - tyA(0, y) - txA(x, 0)\]

Not D-finite

\[(1 - t(x + \bar{x} + \bar{y} + xy)) xyA(x, y) = xy - tyA(0, y) - txA(x, 0)\]

But why?
2. The group of the model

Example. Take $S = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$P(x, y) = \bar{x} + y + x\bar{y}$$
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Observation: $P(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{y}).$$
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They are involutions, and generate a finite dihedral group $G$:

$$
\begin{align*}
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\end{align*}
$$

Remark. $G$ can be defined for any quadrant model with small steps.
Example. Take $S = \{\overline{10}, 01, 1\overline{1}\}$, with step polynomial
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\[ \begin{array}{ccc}
\Phi & \Psi & \Phi \\
(\bar{x}y, y) & (\bar{x}y, \bar{x}) & (\bar{y}, \bar{x}) \\
(\bar{x}y, y) & (\bar{x}y, \bar{x}) & (\bar{y}, \bar{x}) \\
(x, y) & (x, x\bar{y}) & (\bar{y}, x\bar{y}) \\
(x, y) & (x, x\bar{y}) & (\bar{y}, x\bar{y}) \\
\Psi & \Phi & \Psi \\
\end{array} \]

Remark. $G$ can be defined for any quadrant model with small steps.
The group is not always finite

- If \( S = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\} \), then \( P(x, y) = x(1 + y) + y + xy \) and

\[
\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1 + y), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}(1 + \bar{x}))
\]

generate an infinite group:
Example. If \( S = \{01, 10, 11\} \), the orbit of \((x, y)\) is

\[
\begin{align*}
\Phi \quad (\bar{x}y, y) & \quad \Psi \quad (\bar{x}y, \bar{x}) & \quad \Phi \\
(x, y) & \quad \Psi \quad (x, x\bar{y}) & \quad \Phi \\
& \quad \Phi \quad (\bar{y}, x\bar{y}) & \quad \Psi \\
& \quad (\bar{y}, \bar{x})
\end{align*}
\]

and the (alternating) orbit sum is

\[
OS = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}
\]
Classification of quadrant walks with small steps

Theorem

The series $Q(x, y; t)$ is D-finite iff the group $G$ is finite. It is algebraic iff, in addition, the orbit sum is zero.

[[mbm-Mishna 10], [Bostan-Kauers 10], [Kurkova-Raschel 12], [Mishna-Rechnitzer 07], [Melczer-Mishna 13]]

quadrant models: 79

$|G|<\infty$: 23  $|G|=\infty$: 56

D-finite  Not D-finite

OS=0: 4  OS\neq 0: 19

algebraic  transcendental

D-finite  non-singular non-D-finite

singular non-D-finite
Classification of quadrant walks with small steps

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\[ |G| < \infty: 23 \]
\[ |G| = \infty: 56 \]

D-finite

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transcendental

Not D-finite

Random walks in probability

D-finite series
- effective closure properties
- arithmetic properties
- asymptotics
- G-functions

Formal power series algebra

Computer algebra

Complex analysis
\begin{itemize}
\item Properly coloured triangulations ($q$ colours):
\[
T(x, y; t) \equiv T(x, y) = x(q - 1) + xyt \frac{T(x, y) - T(x, 0)}{y} - x^2yt \frac{T(x, y) - T(1, y)}{x - 1}
\]
\end{itemize}
 improper coloured triangulations (\(q\) colours):

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T(x, y; t) \equiv T(x, y) = x(q - 1) + xyt T(x, y) T(1, y) \\
+ xt \frac{T(x, y) - T(x, 0)}{y} - x^2 yt \frac{T(x, y) - T(1, y)}{x - 1}
\]

Isn't this reminiscent of quadrant equations?

\[
Q(x, y; t) \equiv Q(x, y) = 1 + txy Q(x, y) \\
- t \frac{Q(x, y) - Q(0, y)}{x} - t \frac{Q(x, y) - Q(x, 0)}{y}
\]
An old equation [Tutte 73]

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Theorem [Tutte 73-84]

- For $q = 4 \cos^2 \frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies an equation with one catalytic variable $y$. 
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- For $q = 4 \cos^2 \frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies an equation with one catalytic variable $y$. This implies that it is algebraic [mbm-Jehanne 06].
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Theorem [Tutte 73-84]

- For $q = 4 \cos^2 \frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies an equation with one catalytic variable $y$. This implies that it is algebraic [mbm-Jehanne 06].
- For any $q$, the generating function of properly $q$-coloured planar triangulations is differentially algebraic:

\[
2(1 - q)w + (w + 10H - 6wH')H'' + (4 - q)(20H - 18wH' + 9w^2 H'') = 0
\]

with $H(w) = wT(1, 0; \sqrt{w})$. 
In this talk

I. Adapt Tutte’s method to quadrant walks: new and uniform proofs of algebraicity.

II. Extension to an analytic context: some walks with an infinite group (hence not D-finite) are still D-algebraic.
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I. Adapt Tutte’s method to quadrant walks: new and uniform proofs of algebraicity.

II. Extension to an analytic context: some walks with an infinite group (hence not D-finite) are still D-algebraic.
I. New proofs for algebraic models

[In the world of formal power series]
Kreweras’ algebraic model

- The equation (with $\bar{x} = 1/x$ and $\bar{y} = 1/y$):

\[
(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)
= xy - R(x) - S(y)
\]
Kreweras’ algebraic model

- The equation (with $\bar{x} = 1/x$ and $\bar{y} = 1/y$):
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  (1 - t(\bar{x} + \bar{y} + xy)) \cdot xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y) \\
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  \]

- If we take $x = t + ut^2$, both roots of the kernel
  \[
  Y_{0,1} = \frac{x - t \pm \sqrt{(x - t)^2 - 4t^2x^3}}{2tx^2}
  \]

are series in $t$ with rational coefficients in $u$, and can be legally substituted for $y$ in $Q(x, y)$. 

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are series in \( t \) with rational coefficients in \( u \), and can be legally substituted for \( y \) in \( Q(x, y) \). This gives
  \[
  xY_0 = R(x) + S(Y_0), \quad xY_1 = R(x) + S(Y_1),
  \]
so that
  \[
  S(Y_0) - S(Y_1) = xY_0 - xY_1.
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  S(Y_0) - S(Y_1) = xY_0 - xY_1.
  \]

- Are there rational solutions to this equation?
Def. A rational function $D(y; t) \equiv D(y)$ is a decoupling function if, for $Y_{0,1}$ the roots of the kernel,

$$D(Y_0) - D(Y_1) = xY_0 - xY_1.$$
Decoupling functions

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\[
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\]

Example: For Kreweras’ model, \( D(y) = -1/y \) is a decoupling function.

Proof:
\[
\frac{1}{t} = P(x, Y_i) = \frac{1}{x} + \frac{1}{Y_0} + xY_0 = \frac{1}{x} + \frac{1}{Y_1} + xY_1
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Theorem [Bernardi-mbm-Raschel]

- A quadrant model with finite group admits a decoupling function if and only if its orbit sum is zero (exactly 4 models).
- Exactly 9 quadrant models with an infinite group admit a decoupling function.
• The equation

\[ S(Y_0) - S(Y_1) = xY_0 - xY_1, \]

with \( S(y) = tyQ(0, y) \), now reads

\[ S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1), \]

with \( D(y) = -1/y \).
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Are there rational solutions to this equation?
Back to Kreweras’ model

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- Are there rational solutions to this equation?

**Def.** A rational function \( I(y; t) \equiv I(y) \) is an invariant if, the roots \( Y_0, Y_1 \) of the kernel satisfy

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Invariants

**Def.** A rational function $I(y; t) \equiv I(y)$ is an **invariant** if, the roots $Y_0, Y_1$ of the kernel satisfy

$$I(Y_0) = I(Y_1).$$

**Example:** For Kreweras’ model, with kernel $1 - t(\bar{x} + \bar{y} + xy)$, an invariant exists:

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty.$$

**Proof:** check that $I(Y_0) = I(Y_1)$. 

Theorem [Bernardi-mbm-Raschel] A quadrant model admits a rational invariant if and only if the associated group is finite.
Invariants

Def. A rational function \( I(y; t) \equiv I(y) \) is an invariant if, the roots \( Y_0, Y_1 \) of the kernel satisfy

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I(y) = \frac{t}{y^2} - \frac{1}{y} - ty.
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Proof: check that \( I(Y_0) = I(Y_1) \).

Theorem [Bernardi-mbm-Raschel]

A quadrant model admits a rational invariant if and only if the associated group is finite.
Back to Kreweras’ model: combining decoupling functions and invariants

We have

\[ S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1) \quad \text{and} \quad I(Y_0) = I(Y_1) \]

with

\[ S(y) - D(y) = tyQ(0, y) + \frac{1}{y} \quad \text{and} \quad I(y) = \frac{t}{y^2} - \frac{1}{y} - ty. \]
Back to Kreweras’ model: combining decoupling functions and invariants

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\[ S(y) - D(y) = tyQ(0, y) + \frac{1}{y} \] and \[ I(y) = \frac{t}{y^2} - \frac{1}{y} - ty. \]

The invariant lemma

There are few invariants: \( I(y) \) must be a polynomial in \( S(y) - D(y) \) whose coefficients are series in \( t \).
Back to Kreweras’ model: combining decoupling functions and invariants

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\[ S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1) \quad \text{and} \quad I(Y_0) = I(Y_1) \]
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The invariant lemma

There are few invariants: \( I(y) \) must be a polynomial in \( S(y) - D(y) \)
whose coefficients are series in \( t \).

\[ \frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyQ(0, y) + \frac{1}{y} \right)^2 - \left( tyQ(0, y) + \frac{1}{y} \right) + c \]

Expanding at \( y = 0 \) gives the value of \( c \).
Back to Kremeras’ model:
combining decoupling functions and invariants

We have
\[ S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1) \quad \text{and} \quad I(Y_0) = I(Y_1) \]

with
\[ S(y) - D(y) = tyQ(0, y) + \frac{1}{y} \quad \text{and} \quad l(y) = \frac{t}{y^2} - \frac{1}{y} - ty. \]

The invariant lemma

There are few invariants: \( l(y) \) must be a polynomial in \( S(y) - D(y) \)
whose coefficients are series in \( t \).

\[
\frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyQ(0, y) + \frac{1}{y} \right)^2 - \left( tyQ(0, y) + \frac{1}{y} \right) - 2t^2 Q(0, 0).
\]

Expanding at \( y = 0 \) gives the value of \( c \).
Algebraic models: a uniform approach

All models with a finite group and a zero orbit sum have a rational invariant and a decoupling function $\Rightarrow$ uniform solution via the solution of an equation with one catalytic variable

\[ \begin{array}{cccc}
\downarrow & \rightarrow & \uparrow & \rightarrow \\
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\end{array} \]
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This applies as well to weighted algebraic models [Kauers, Yatchak 14(a)]:
II. Infinite groups: some differentially algebraic models

[An excursion in the world of analytic functions]

Fayolle, Iasnogorodski, Malyshev [1999]
The role of decoupling functions

**Theorem [Bernardi-mbm-Raschel]**

For the 9 models with an infinite group and a decoupling function, the series $Q(x, y; t)$ is D-algebraic. That is, it satisfies a DE in $t$ (and a DE in $x$, and a DE in $y$) with polynomial (or even constant) coefficients.
A weaker (and analytic) notion of invariants

- Still require that $I(Y_0) = I(Y_1)$, where $Y_0, Y_1$ are the roots of the kernel ... but only for some values of $x$ (and $t$).
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- meromorphicity condition in a domain
Can we find weak invariants?

**Theorem** [Fayolle et al. 99, Raschel 12]

For each non-singular model, there exists an (explicit) weak invariant of the form

\[ I(y; t) = \wp(\mathcal{R}(y; t), \omega_1(t), \omega_3(t)) \]

where

- \( \wp \) is Weierstrass elliptic function
- its periods \( \omega_1 \) and \( \omega_3 \) are elliptic integrals
- its argument \( \mathcal{R} \) is also an elliptic integral
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\[
\omega_1 = i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-\delta(x)}}, \quad \omega_3 = \int_{x(y_1)}^{x_1} \frac{dx}{\sqrt{\delta(x)}}.
\]

\[
\mathcal{R}(y; t) = \int_{f(y_2)}^{f(y)} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}
\]

\(g_2, g_3\) polynomials in \(t\), \(f(y)\) rational in \(y\) and algebraic in \(t\).
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**Proposition** [Bernardi-mbm-Raschel]

\( I(y; t) \) is D-algebraic in \( y \) and \( t \).
Combining decoupling functions and invariants

For a model with decoupling function $D(y)$ we have, for $x \in (x_1, x_2)$:

$$S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1)$$

and

$$I(Y_0) = I(Y_1)$$

where $S(y) = K(0, y)Q(0, y)$ and $I(y)$ is the weak invariant.
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The invariant lemma [Litvinchuk 00]

There are few invariants: $S(y) - D(y)$ must be a rational function in $I(y)$. The value of this rational function is found by looking at the poles and zeroes of $S(y) - D(y)$. 
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Example: \( \square \) is decoupled with \( D(y) = -1/y \) and

\[
S(y) + \frac{1}{y} = t(1 + y)Q(0, y) + \frac{1}{y} = \frac{l'(0)}{I(y) - I(0)} - \frac{l'(0)}{I(-1) - I(0)} - 1
\]
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**Corollary**

For the 9 models with an infinite group and a decoupling function, the series $Q(x, y; t)$ is D-algebraic.
quadrant models: 79

\[ |G| < \infty : 23 \quad |G| = \infty : 56 \]

D-finite

D-finite

\[ \text{OS} = 0 : 4 \quad \text{OS} \neq 0 : 19 \]

algebraic

transcendental

Not D-finite
quadrant models: 79

\[ |G| < \infty: \ 23 \]
\[ |G| = \infty: \ 56 \]

D-finite

- \( \text{dec. 4} \)
- \( \text{no dec. 19} \)

algebraic

transcendental

Not D-finite

- \( \text{dec. 9} \)
- \( \text{no dec. 47} \)

D-algebraic

???

To do:

- find explicit DEs (done for \( y \))
- Nature of \( Q(x, y; t) \) when no decoupling function exists?
  [Dreyfus, Hardouin, Roques, Singer 17(a)]