A length function for the complex reflection group $G(r, r, n)$

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General Definitions

- S_n is the symmetric group on $\{1, \ldots, n\}$.
- $\mathbb{Z}_r$ is the cyclic group of order r.
- ζ_r is the primitive $r - th$ root of unity.

Complex reflection groups

- $G(r, n) =$ group of all matrices $\pi = (\sigma, k)$, where:
- $\sigma = a_1 \cdots a_n \in S_n$.
- $k = (k_1, \ldots, k_n) \in \mathbb{Z}_r^n$. ($k$-vector)
- $\pi = (\sigma, k)$ is the $n \times n$ monomial matrix with non-zero entries $\zeta_r^{k_i}$ in the (a_i, i) positions.

Example

$(n = 3, r = 4)$

$$\pi(312, (1, 3, 3)) = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{pmatrix}$$

- For $p|r$, $G(r,p,n)$ is the subgroup of $G(r,n)$ consisting of matrices (σ, k) satisfying

$$\prod_{i=1}^{n} (\zeta_r^{k_i})^{\frac{r}{p}} = 1.$$

- Hence $G(r,r,n)$ is the group of such matrices satisfying:

$$\prod_{i=1}^{n} (\zeta_r^{k_i}) = 1$$

One-line notation

We denote an element of $G(r, p, n)$ in a more concise manner:

$$(\sigma, k) = a_1^{k_1} \cdots a_n^{k_n}$$

for $\sigma = a_1 \cdots a_n$ and $k = (k_1, \dots, k_n)$.

Example

$$\pi(312, (1, 3, 3)) = 3^1 1^3 2^3$$

Our goal

Various sets of generators have been defined for complex reflection groups but (as far as we know), no length function has been formulated.
We provide such a function for the case of $G(r, r, n)$ with a specific choice of generating set proposed by Shi.

Shi's Generators for $G(r, r, n)$

- For each $i \in \{1, \dots, n-1\}$ let $s_i = (i, i+1)$ be the familiar adjacent transpositions generating S_n.
- Define $t_0 = (1^{r-1}, n^1)$.

Theorem

The set $\{t_0, s_1, \dots, s_{n-1}\}$ generates $G(r, r, n)$.

Example of generators acting from the right

Applying s_1 from the right:

$$\pi = 3^0 2^2 1^{-1} 4^{-1} \mapsto 2^2 3^0 1^{-1} 4^{-1}$$

Applying t_0 from the right:

$$\pi = 2^0 1^2 3^{-1} 4^{-1} \mapsto 4^{-2} 1^2 3^{-1} 2^1$$

Remark

Places are exchanged, the $k-$ vector is not preserved.

Example of generators acting from the left

Applying s_1 from the left:

$$\pi = 2^0 1^2 3^{-1} 4^{-1} \mapsto 1^0 2^2 3^{-1} 4^{-1}$$

Applying t_0 from the left:

$$\pi = 2^0 1^2 3^{-1} 4^{-1} \mapsto 2^0 4^2 3^{-1} 1^{-1}$$

Remark

Numbers are exchanged and the k-vector is preserved.

The affine group

The affine Weyl group $\tilde{S}_n$ is defined as follows:

$$\tilde{S}_n = \{w : \mathbb{Z} \to \mathbb{Z} \mid w(i+n) = w(i)+n, \forall i \in \{1, \ldots, n\}, \sum_{i=1}^{n} w(i) = \binom{n+1}{2}\}.$$

Each affine permutation can be written in *integer window notation* in the form:

$$\pi = (\pi(1), \ldots, \pi(n)) = (b_1, \ldots, b_n).$$

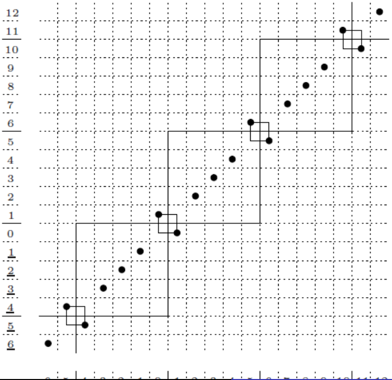
By writing $b_i = n \cdot k_i + a_i$, we can use the *residue window notation*:

$$\pi = a_1^{k_1} \cdots a_n^{k_n}.$$

where $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$.

Generators for the affine group

- For each $i \in \{1, \ldots, n-1\}$ let $s_i = (i, i+1)$ be the known adjacent transpositions generating S_n.
- Define $s_0 = (1, n^{-1})$.



Theorem

Let $\pi = a_1^{k_1} \cdots a_n^{k_n} \in \tilde{S}_n$. *Then*

$$\ell(\pi) = \sum_{\substack{1 \le i < j \le n \\ a_i < a_j}} |k_j - k_i| + \sum_{\substack{1 \le i < j \le n \\ a_i > a_j}} |k_j - k_i - 1|$$

Example

If $\pi = 3^{-1}1^0 4^1 2^0$ then:

$$\ell(\pi) = |1-(-1)| + |1-0| + |0-(-1)-1| + |0-(-1)-1| + |0-1-1| = 5$$

Another presentation of $\tilde{S}_n$

Each affine permutation $\pi = a_1^{k_1} \cdots a_n^{k_n}$ can also be written as a monomial matrix:

$$M_\pi = (m_{ij}) = \begin{cases} 0 & i \neq \sigma(j) \\ x^{k_i} & i = \sigma(j) \end{cases}$$

Example

$(n = 4)$

$$\pi = 3^{-1}1^0 4^1 2^0 = \begin{pmatrix} 0 & x^0 & 0 & 0 \\ 0 & 0 & 0 & x^0 \\ x^{-1} & 0 & 0 & 0 \\ 0 & 0 & x^1 & 0 \end{pmatrix}$$

Mapping $\tilde{S}_n$ to $G(r,r,n)$

- Shi defines a homomorphism $\eta : \tilde{S}_n \to G(r,r,n)$ by substituting a primitive r-th root of unity ζ_r in place of x.
- He tried to adapt his length function for the affine groups to the case of $G(r,r,n)$ but did not obtain a closed formula.
- Here we provide such a formula.

Difficulties in adapting Shi's formula

In $G(r, r, n)$ each element does not have a uniquely defined k-vector, as adding a multiple of r to any k_i does not change π as an element of $G(r, r, n)$.

Example

The permutations $4^5 2^{-4} 3^{-2} 1^1$ and $4^0 2^{-4} 3^3 1^1$ represent the same element of $G(5, 5, 4)$.

The normal form

Definition

A permutation $(p, k^0) \in G(r, r, n)$ *is said to be in* **normal form** *if the following conditions are met:*

1. $\sum_{i=1}^{n} k_i^0 = 0$
2. $|max(k^0) - min(k^0)| \leq r$
3. *If there exist* $i < j$ *such that* $|k_j^0 - k_i^0| = r$ *then* $k_j^0 - k_i^0 = r$.

If (p, k^0) *is in normal form and is equivalent to* (p, k) *then we say that* (p, k^0) *is a normal form of* (p, k).

Example

The normal form of $4^{-8}1^{15}3^{12}2^{9} \in G(7,7,4)$ is $4^{-1}1^{1}3^{-2}2^{2}$.

Theorem

- *For each $\pi \in G(r,r,n)$ a normal form exists and is unique.*
- *Shi's length function, when applied to all representatives of a permutation in $G(r,r,n)$, attains its minimum on the normal form representative.*

Decomposition Into Right Cosets of S_n

- Let $\pi = (k, \sigma) \in G(r, r, n)$.
- As we have seen, for each generator τ of S_n , π and $\tau\pi$ have the same k-vector.
- Hence, it is natural and straightforward to decompose $G(r, r, n)$ into right cosets.
- Each right coset has a unique representative $\pi = (k, \sigma)$ which has minimal length.
- This leads us to a new length function for $G(r, r, n)$.

The length function for $G(r, r, n)$

Let $\pi = a_1^{k_1} \cdots a_n^{k_n} \in G(r, r, n)$.
Write $\pi = u \cdot \sigma$ where $u \in S_n$ and σ is the minimal length representative. Then:

Theorem

$$\ell(\pi) = \sum_{1 \leq i < j \leq n} |k_j - k_i| - noninv(k) + inv(u)$$

where

$$noninv(k) = \#\{(i, j) \mid i < j, k(i) < k(j)\}$$

and (as usual)

$$inv(u) = \#\{(i, j) \mid i < j, u(i) > u(j)\}.$$

Length Example

Let $\pi = 3^1 1^{-2} 2^0 4^1 \in G(4,4,4)$.
Then $\sigma = 1^1 4^{-2} 3^0 2^1$, and $u = |\pi||\sigma|^{-1} = 3421$.
Hence:

$$\sum_{1 \leq i < j \leq n} |k_j - k_i| = |-2-1|+|0-1|+|1-1|+|0-(-2)|+|1-(-2)|+|1-0| = 10$$

And:

$$noninv(k) = 3$$

while

$$inv(u) = 5$$

so that $\ell(\pi) = 10 - 3 + 5 = 12$

Finding the minimal-length representative

The minimal-length element $\sigma = a_1^{k_1} \cdots a_n^{k_n} \in G(r, r, n)$ for the k-vector $(k_1, \dots, k_n)$
(abbreviated $a_1 a_2 \cdots a_n \in S_n$)
is the unique one with the following property:
$a_i < a_j$ iff:

- $k(i) > k(j)$, or
- $k(i) = k(j)$ and $i < j$

Example

If $k = (-2, 1, -1, 1, 2, -1)$ then $\sigma = 624315$

Open question: What is the generating function?

Let $G_{r,r,n}(q) = \sum_{\pi \in G_{r,r,n}} q^{\ell(\pi)}$.

From the coset decomposition it is clear that $G_{r,r,n}(q)$ has $[n]_q!$ as a factor.

Example

$$G_{4,4,4}(q) = [4]_q!(1+2q^2+3q^3+4q^4+5q^5+7q^6+8q^7+10q^8+12q^9+7q^{10}+3q^{11})$$

$$G_{6,6,3}(q) = [3]_q!(1+q+2q^2+2q^3+3q^4+3q^5+4q^6+4q^7+5q^8+5q^9+6q^{10})$$

A possible direction...

- There is a bijection between left cosets of S_n in the affine group and certain types of partitions (see Bjorner and Brenti (1996) and Eriksson and Eriksson (1998)).
- In B-B, each partition is the *inversion table* of the corresponding left coset (i.e., of its ascending minimal-length representative).
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- This correspondence yields the following generating function for length in the affine group:

$$\tilde{S}_n(q) = \frac{[n]_q!}{(1-q)(1-q^2)\cdots(1-q^n)}$$

- A similar approach may work in our case of right cosets in $G(r, r, n)$.

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Thank you!!