\( \nu \)-Tamari lattices via subword complexes

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In this talk

**Theorem**

*The *ν*-Tamari lattice is the dual of a well chosen subword complex.*

![Diagram of the ν-Tamari lattice and subword complexes](image)
In this talk

**Theorem**

*The ν-Tamari lattice is the dual of a well chosen subword complex.*

The picture actually contains three theorems and one corollary. Please remember the picture!
Tamari lattices

The Tamari-lattice: partial order on Catalan objects.

Tamari lattices

The Tamari-lattice is a partial order on Catalan objects. Covering relation:

Rotation on binary trees
The Tamari-lattice is a partial order on Catalan objects. Covering relation:

Interchanging operation on Dyck paths
Motivated by trivariate diagonal harmonics, F. Bergeron introduced the $m$-Tamari lattice on Fuss-Catalan paths.

$m$-Tamari lattices: nice enumerative properties

- Number of elements: Fuss Catalan number $\frac{1}{mn+1} \binom{(m+1)n}{n}$

- Number of intervals: $\frac{m+1}{n(mn+1)} \binom{(m+1)^2n+m}{n-1}$


- Number of “decorated” intervals: $(m + 1)^n(mn + 1)^{n-2}$


Conjecture (F. Bergeron (Haiman for $m = 1$))

The number of intervals is conjecturally interpreted as the dimension of the alternating component of a space in trivariate diagonal harmonics. Decorated intervals correspond to the entire space.
$m$-Tamari lattices: nice geometry

The 2-Tamari lattice for $n = 4$

C.–Padrol–Sarmiento, 2016: The Hasse diagram of $m$-Tamari lattices are the edge graphs of (tropical) polytopal subdivisions of associahedra.
\(\nu\)-Tamari lattices

Préville-Ratelle–Viennot:
Introduced the \(\nu\)-Tamari lattice on lattice paths weakly above \(\nu\).

Covering relation:

\[
\begin{array}{c}
\text{Diagram 1} \quad \text{Diagram 2} \\
\begin{array}{c}
1 \\
0 \\
1 \\
2 \\
\end{array} \quad \begin{array}{c}
1 \\
0 \\
q \\
p \\
\end{array}
\end{array}
\]

\[<_{\nu}\]

Theorem (Préville-Ratelle–Viennot)
This partial order defines a lattice structure on \(\nu\)-Dyck paths.

Préville-Ratelle–Viennot: Introduced the $\nu$-Tamari lattice on lattice paths weakly above $\nu$.

Covering relation:

They also have nice enumerative and geometric properties.


Theorem 1

The Hasse diagram of the \( \nu \)-Tamari lattice is the facet adjacency graph of a well chosen subword complex. 

This generalizes a known result by Woo (2004), Pilaud–Pocchiola (2010), Stump (2010), and Stump-Serrano (2010) in the classical case.
Subword complexes

\[ W = \mathfrak{S}_{n+1} \] group of permutations of \([n + 1]\]

\[ S = \{s_1, \ldots, s_n\} \] the set of simple generators \(s_i = (i \ i + 1)\)

\[ Q = (q_1, \ldots, q_m) \] a word in \(S\)

\[ \pi \in W \]
Subword complexes

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\[ Q = (q_1, \ldots, q_m) \text{ a word in } S \]
\[ \pi \in W \]

Definition (Knutson–Miller, 2004)

The **subword complex** \( \Delta(Q, \pi) \) is the simplicial complex whose

\[ \text{faces } \longleftrightarrow \text{ subwords } P \text{ of } Q \text{ such that } Q \setminus P \text{ contains a reduced expression of } \pi \]

Subword complexes - Example modify $s_3$

In type $A_2$:

$W = S_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$
Subword complexes - Example modify $s_3$

In type $A_2$:

$W = S_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

$Q = \begin{pmatrix} s_1, s_2, s_1, s_2, s_1 \\ q_1, q_2, q_3, q_4, q_5 \end{pmatrix}$ and $\pi = [3\ 2\ 1]$
Subword complexes - Example modify $s_3$

In type $A_2$:

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$Q = (s_1, s_2, s_1, s_2, s_1)$ and $\pi = [3 \ 2 \ 1]$

$\Delta(Q, \pi)$ is isomorphic to

$q_2$

$q_3$

$q_4$

$q_5$
Subword complexes - Example modify $s_3$

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Subword complexes - Example modify $s_3$

In type $A_2$:

$\mathcal{W} = S_3$, $S = \{s_1, s_2\} = \{(1, 2), (2, 3)\}$

$Q = (s_1, s_2, s_1, q_2, q_3, q_4, q_3, q_2)$ and $\pi = [3, 2, 1] = s_1 s_2 s_1$

$\Delta(Q, \pi)$ is isomorphic to
Subword complexes - Example modify \( s_3 \)

In type \( A_2 \):

\( \mathcal{W} = S_3, \ S = \{ s_1, s_2 \} = \{ (1 2), (2 3) \} \)

\( Q = ( s_1, s_2, , s_1, q_3, q_4, ) \) and \( \pi = [3 2 1] = s_1 s_2 s_1 \)

\( \Delta(\mathcal{W}, \pi) \) is isomorphic to
Subword complexes - Example modify $s_3$

In type $A_2$:

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$Q = (s_1, s_2, s_1, \ldots, q_4, q_5)$ and $\pi = [3 2 1] = s_1 s_2 s_1$

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Subword complexes - Example modify $s_3$

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Subword complexes - Example modify $s_3$

In type $A_2$:

$W = S_3, \ S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

$Q = (s_1, s_2, s_1, s_2, s_1)_{q_1, q_2, q_3, q_4, q_5}$ and $\pi = [3\ 2\ 1] = s_1 s_2 s_1 = s_2 s_1 s_2$

$\Delta(Q, \pi)$ is isomorphic to
The subword complex result

Theorem 1

The Hasse diagram of the $\nu$-Tamari lattice is the facet adjacency graph of a well chosen subword complex $\Delta(Q_\nu, \pi_\nu)$.

\[ Q_\nu = (s_3, s_2, s_1, s_4, s_3, s_2, s_4, s_3, s_5, s_4) \]
\[ \pi_\nu = [1, 4, 3, 5, 2, 6] \]
The subword complex result

These objects keep showing up in independent places:


They are special but still somewhat mysterious.
Facets and $\nu$-trees

The facets of $\Delta(Q_\nu, \pi_\nu)$ are given by $\nu$-trees. Two facets are adjacent $\iff$ the trees are related by rotation.

$s_2 s_3 s_2 s_4 = [1, 4, 3, 5, 2, 6]$ rotation $s_3 s_2 s_3 s_4 = [1, 4, 3, 5, 2, 6]$
Facets and \( \nu \)-trees

The facets of \( \Delta(Q_\nu, \pi_\nu) \) are given by \( \nu \)-trees. Two facets are adjacent \( \iff \) the trees are related by rotation.

\[
s_2s_3s_2s_4 = [1, 4, 3, 5, 2, 6]
\]

\( \nu \)-tree:
(Serrano–Stump) Maximal sets of lattice points above \( \nu \) avoiding north-east increasing chains \( p, q \) such that \( p \perp q \) is above \( \nu \).
(This talk) some “maximal” binary trees fitting above \( \nu \).
The rotation lattice of $\nu$-trees

Theorem 1 follows from:

**Theorem 2**

*The $\nu$-Tamari lattice is isomorphic to the rotation lattice on $\nu$-trees.*
The rotation lattice of $\nu$-trees

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**Theorem 2**

*The $\nu$-Tamari lattice is isomorphic to the rotation lattice on $\nu$-trees.*
The lattice of $\nu$-bracket vectors

The meet and join: very simple on $\nu$-trees.
The lattice of $\nu$-bracket vectors

The meet and join: very simple on $\nu$-trees.

**Theorem 3**

*The $\nu$-Tamari lattice is isomorphic to the lattice of $\nu$-bracket vectors under componentwise order.*

$b(T) = \text{read } y\text{-coordinates of the nodes in in-order.}$
The lattice of $\nu$-bracket vectors

$\nu$-bracket vectors are easily characterized. Their meet is obtained by taking componentwise minimum.

Corollary

*Simple proof of the lattice property.*
In summary

Thm 1 & Thm 2

Thm 3 & Cor
Multi $\nu$-Tamari complexes

For $k \geq 1$, define the $(k, \nu)$-Tamari complex

faces $\leftrightarrow$ sets of points above $\nu$ avoiding $(k + 1)$-north-east incr. chains.

- $\nu = (NE)^n$: simplicial multiassociahedron $\Delta_{n+2,k}$.
  Conjectured to be realizable as a polytope (Jonsson 2004).

- $k = 1$, $\nu$ without consecutive north steps: facet adjacency graph = edge graph of a polytopal subdivision of an associahedron.

Question

Is the facet adjacency graph of the $(k, \nu)$-Tamari complex the edge graph of a polytopal subdivision of a multiassociahedron?
Multi $\nu$-Tamari complexes

**Proposition**

Let $m \geq k$ and $\nu = (NE^m)^{k+1}$. The facet adjacency graph $G_{k,\nu}$ of the Fuss-Catalan $(k, \nu)$-Tamari complex is the edge graph of a polytopal subdivision of the multi-associahedron $\Delta_{2k+2,k}$.

$k = 2$ and $\nu = (NE^5)^3$

$k = 3$ and $\nu = (NE^5)^4$

$\Delta_{2k+2,k}$: a $k$-dimensional simplex

Subdivision: *staircase subdivision* of its $(m - k + 1)$ dilation.
My birthday present!

Is this true in general?
Merci!