The 0-Rook Monoid

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Symmetric Group and Rook Monoid

\[ S_n \]

Symmetric Group
The 0-Rook Monoid

Symmetric Group and Rook Monoid

\[ \mathfrak{S}_n \xrightarrow{q=1} \mathcal{H}_n(q) \]

Symmetric Group \hspace{2cm} Iwahori-Hecke algebra

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The 0-Rook Monoid
\[ \mathfrak{S}_n \xrightarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H_n^0 \]

- Symmetric Group
- Iwahori-Hecke algebra
- Hecke monoid at \( q = 0 \)
$\mathfrak{S}_n \overset{q=1}{\leftarrow} \mathcal{H}_n(q) \overset{q=0}{\rightarrow} H^0_n$

Symmetric Group

Iwahori-Hecke algebra

Hecke monoid at $q = 0$

\[ s_i^2 = 1 \]
\[ s_{i+1}s_is_{i+1} = s_is_{i+1}s_i \]
\[ s_is_j = s_js_i \]

\[ T_i^2 = q1 + (q - 1)T_i \]
\[ T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i \]
\[ T_iT_j = T_jT_i \]

\[ \pi_i^2 = \pi_i \]
\[ \pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i \]
\[ \pi_i\pi_j = \pi_j\pi_i \]
Symmetric Group and Rook Monoid

\[ \mathcal{S}_n \xleftarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H^0_n \]

Symmetric Group

Iwahori-Hecke algebra

Hecke monoid at \( q = 0 \)

\[ s_i^2 = 1 \]
\[ s_{i+1}s_i s_{i+1} = s_i s_{i+1} s_i \]
\[ s_i s_j = s_j s_i \]

\[ T_i^2 = q1 + (q - 1) T_i \]
\[ T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i \]
\[ T_i T_j = T_j T_i \]

\[ \pi_i^2 = \pi_i \]
\[ \pi_{i+1} \pi_i \pi_{i+1} = \pi_i \pi_{i+1} \pi_i \]
\[ \pi_i \pi_j = \pi_j \pi_i \]
\[ \mathcal{S}_n \leftarrow q=1 \mapsto \mathcal{H}_n(q) \rightarrow q=0 \rightarrow H^0_n \]

Symmetric Group

\[
\begin{align*}
    s_i^2 &= 1 \\
    s_{i+1}s_is_{i+1} &= s_is_{i+1}s_i \\
    s_is_j &= s_js_i
\end{align*}
\]

\[ s_i = T_i \]

Iwahori-Hecke algebra

\[
\begin{align*}
    T_i^2 &= q1 + (q - 1)T_i \\
    T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i \\
    T_iT_j &= T_jT_i
\end{align*}
\]

Hecke monoid at \( q = 0 \)

\[
\begin{align*}
    \pi_i^2 &= \pi_i \\
    \pi_{i+1}\pi_i\pi_{i+1} &= \pi_i\pi_{i+1}\pi_i \\
    \pi_i\pi_j &= \pi_j\pi_i \\
    \pi_i &= T_i + 1
\end{align*}
\]
Interesting properties of $H_n^0$

- $|H_n^0| = n!$
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- $|H_n^0| = n!$
- $H_n^0$ acts on $\mathfrak{S}_n$ (bubble sort):
  $$\pi_5 \cdot 3726145 = 3726415$$
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Interesting properties of $H_n^0$

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  \]
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  \]
- An element of $H_n^0$ is characterized by its action on the identity.
- Its simple and projective modules are well-known and combinatorial [Norton-Carter].
Interesting properties of $H^0_n$

- $|H^0_n| = n!$
- $H^0_n$ acts on $\mathcal{S}_n$ (bubble sort):
  \[
  \pi_5 \cdot 3726145 = 3726415 \\
  \pi_5 \cdot 3726415 = 3726415
  \]
- An element of $H^0_n$ is characterized by its action on the identity.
- Its simple and projective modules are well-known and combinatorial [Norton-Carter].
- The induction and restriction of modules gives us a structure of tower of monoids, linked to QSym and NCSF [Krob-Thibon].
**The rook monoid**

**Rook matrix** of size $n = \text{set of non attacking rooks on an } n \times n$ matrix.
The rook monoid

**Rook matrix** of size $n = \text{set of non attacking rooks on an } n \times n$ matrix.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & \\
0 & 0 & \\
0 & 0 & 0 & \\
0 & \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & \\
0 & 0 & \\
0 & 0 & 0 & \\
0 & \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

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The rook monoid

**Rook matrix** of size \( n \) = set of non-attacking rooks on an \( n \times n \) matrix.

<table>
<thead>
<tr>
<th>Rook Matrix</th>
<th>Rook Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0 0</td>
<td>0 4 2 3 1</td>
</tr>
<tr>
<td>0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 0</td>
<td></td>
</tr>
</tbody>
</table>

The product of two rook matrices is a rook matrix.

**Rook Monoid** \( R_n \) = submonoid of the rook matrices \( M_n \) ⊃ \( R_n \) ⊃ \( S_n \)
The rook monoid

**Rook matrix** of size $n = \text{set of non attacking rooks on an } n \times n$ matrix.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Rook Matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Rook Vector

05 4 2 3 1

05 3 02 4 1
The rook monoid

**Rook matrix** of size $n =\) set of non attacking rooks on an $n \times n$ matrix.

Rook Matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Rook Vector
\[
0_5 \ 4 \ 2 \ 3 \ 1
\]

The product of two rook matrices is a rook matrix.

**Rook Monoid** $R_n = \text{submonoid of the rook matrices}$

$M_n \supset R_n \supset \mathfrak{S}_n$
Symmetric Group and Rook Monoid

The 0-Rook Monoid

Representation theory

Work in Progress

\[ S_n \]

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The 0-Rook Monoid
\[ \mathfrak{S}_n \overset{q=1}{\longrightarrow} \mathcal{H}_n(q) \]

Iwahori
\[ \mathcal{S}_n \xleftarrow{\ q=1 \ } \mathcal{H}_n(q) \xrightarrow{\ q=0 \ } H_0^n \]

Iwahori
\[ \mathcal{S}_n \xrightarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H_n^0 \]

\[ R_n \]

Iwahori
The symmetric group $\mathfrak{S}_n$ and the rook monoid $R_n$ have representations in the Iwahori-Hecke algebra $H_n(q)$ and its Iwahori subalgebra $I_n(q)$, respectively. The Iwahori-Hecke algebra is a deformation of the group algebra of the symmetric group. Iwahori and Solomon extensively studied these structures.
Symmetric Group and Rook Monoid

The 0-Rook Monoid

Representation theory

Work in Progress

\[
\mathfrak{S}_n \quad \xrightarrow{q=1} \quad \mathcal{H}_n(q) \quad \xrightarrow{q=0} \quad \mathcal{H}_n^0
\]

\[
R_n \quad \xleftarrow{q=1} \quad \mathcal{I}_n(q) \quad \xrightarrow{q=0} \quad ??
\]

Iwahori, Solomon.
Symmetric Group and Rook Monoid

$\mathcal{S}_n \xleftarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H_n^0$

$I_n(q) \xleftarrow{q=1} \mathcal{I}_n(q) \xrightarrow{q=0} R_n^0$

Iwahori, Solomon.
Contents

1 Symmetric Group and Rook Monoid

2 The 0-Rook Monoid

3 Representation theory

4 Work in Progress
Definition by right action on $R_n$

Operators $\pi_0, \pi_1, \ldots \pi_{n-1}$ acting on rook vectors
Definition by right action on $R_n$

Operators $\pi_0, \pi_1, \ldots, \pi_{n-1}$ acting on rook vectors

Bubble sort operators $\pi_1, \ldots, \pi_{n-1}$:

$$(r_1 \ldots r_n) \cdot \pi_i = \begin{cases} 
  r_1 \ldots r_{i-1} r_{i+1} r_i r_{i+2} \ldots r_n & \text{if } r_i < r_{i+1}, \\
  r_1 \ldots r_n & \text{otherwise},
\end{cases}$$

Deletion operator $\pi_0$:

$$(r_1 \ldots r_n) \cdot \pi_0 = 0 r_2 \ldots r_n.$$
Definition by right action on $R_n$

Operators $\pi_0, \pi_1, \ldots \pi_{n-1}$ acting on rook vectors

Bubble sort operators $\pi_1, \ldots, \pi_{n-1}$:

$$(r_1 \ldots r_n) \cdot \pi_i = \begin{cases} r_1 \ldots r_{i-1}r_{i+1}r_ir_{i+2} \ldots r_n & \text{if } r_i < r_{i+1}, \\ r_1 \ldots r_n & \text{otherwise,} \end{cases}$$

Deletion operator $\pi_0$:

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Deletion operator $\pi_0$

$$(r_1 \ldots r_n) \cdot \pi_0 = 0r_2 \ldots r_n.$$
Definition by presentation

Generators: $\pi_0, \ldots, \pi_{n-1}$

Relations:

- $\pi_i^2 = \pi_i$ for $0 \leq i \leq n-1,$
- $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for $1 \leq i \leq n-2,$
- $\pi_i \pi_j = \pi_j \pi_i$ for $|i-j| > 1.$
- $\pi_1 \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 \pi_1$
Definition by presentation

Generators: \( \pi_0, \ldots, \pi_{n-1} \)

Relations:

\[\pi_i^2 = \pi_i \quad 0 \leq i \leq n - 1,\]

\[\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \quad 1 \leq i \leq n - 2,\]

\[\pi_i \pi_j = \pi_j \pi_i \quad |i - j| > 1.\]

\[\pi_1 \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 \pi_1\]
Theorem

Both definitions (presentation and action on $R_n$) are equivalent.
Theorem

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Key Fact:

Theorem

The map \( f : R_n^0 \rightarrow R_n \) \( r \mapsto 1_n \cdot r \) is a bijection.
Symmetric Group and Rook Monoid

The 0-Rook Monoid

Representation theory

Work in Progress

Theorem

Both definitions (presentation and action on \( R_n \)) are equivalent.

Key Fact:

The map \( f : \left\{ \begin{array}{l} R_0^n \longrightarrow R_n \\ r \mapsto 1_n \cdot r \end{array} \right. \) is a bijection.

This also gives us canonical reduced expression of elements of \( R_n^0 \).

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The 0-Rook Monoid
### Canonical reduced expression

**Example**: using coset $R_5^0 / R_4^0$

<table>
<thead>
<tr>
<th>Coset</th>
<th>30145</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_5^0$</td>
<td>30145</td>
</tr>
<tr>
<td>$R_4^0$</td>
<td></td>
</tr>
<tr>
<td>$R_5^0 / R_4^0$</td>
<td>30145</td>
</tr>
</tbody>
</table>

---

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The 0-Rook Monoid
Canonical reduced expression

<table>
<thead>
<tr>
<th>Example: using coset $R_5^0 / R_4^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$30145$</td>
</tr>
<tr>
<td>$\downarrow \pi_4$</td>
</tr>
<tr>
<td>$30154$</td>
</tr>
<tr>
<td>$30143$</td>
</tr>
<tr>
<td>$\downarrow \pi_3$</td>
</tr>
<tr>
<td>$30514$</td>
</tr>
<tr>
<td>$\downarrow \pi_2$</td>
</tr>
<tr>
<td>$35014$</td>
</tr>
<tr>
<td>$\downarrow \pi_1$</td>
</tr>
<tr>
<td>$53014$</td>
</tr>
<tr>
<td>$\downarrow \pi_0$</td>
</tr>
<tr>
<td>$03014$</td>
</tr>
<tr>
<td>$\odot \pi_2$</td>
</tr>
<tr>
<td>$30014$</td>
</tr>
</tbody>
</table>
Canonical reduced expression

Example: using coset $R_5^0 / R_4^0$

\[
\begin{align*}
30145 \\
\downarrow \pi_4 \\
30154 \\
\downarrow \pi_3 \\
30514
\end{align*}
\]
### Canonical reduced expression

**Example: using coset $R_5^0 / R_4^0$**

<table>
<thead>
<tr>
<th>Step</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30145</td>
</tr>
<tr>
<td>2</td>
<td>[\downarrow \pi_4]</td>
</tr>
<tr>
<td>3</td>
<td>30154</td>
</tr>
<tr>
<td>4</td>
<td>[\downarrow \pi_3]</td>
</tr>
<tr>
<td>5</td>
<td>30514</td>
</tr>
<tr>
<td>6</td>
<td>[\downarrow \pi_2]</td>
</tr>
<tr>
<td>7</td>
<td>35014</td>
</tr>
</tbody>
</table>
### Canonical reduced expression

**Example: using coset $R_5^0 / R_4^0$**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Corresponding π</th>
</tr>
</thead>
<tbody>
<tr>
<td>30145</td>
<td>π_4</td>
</tr>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>30154</td>
<td>π_3</td>
</tr>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>30514</td>
<td>π_2</td>
</tr>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>35014</td>
<td>π_1</td>
</tr>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>53014</td>
<td></td>
</tr>
</tbody>
</table>

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## Canonical reduced expression

### Example: using coset $R_5^0 / R_4^0$

<table>
<thead>
<tr>
<th>Expression</th>
<th>Coset</th>
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<tr>
<td>$30145$</td>
<td>$R_5^0$</td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>$30154$</td>
<td>$R_4^0$</td>
</tr>
<tr>
<td>$\downarrow \pi_3$</td>
<td></td>
</tr>
<tr>
<td>$30514$</td>
<td></td>
</tr>
<tr>
<td>$\downarrow \pi_2$</td>
<td></td>
</tr>
<tr>
<td>$35014$</td>
<td></td>
</tr>
<tr>
<td>$\downarrow \pi_1$</td>
<td></td>
</tr>
<tr>
<td>$53014$</td>
<td></td>
</tr>
<tr>
<td>$\downarrow \pi_0$</td>
<td></td>
</tr>
<tr>
<td>$03014$</td>
<td></td>
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<thead>
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<tbody>
<tr>
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</tr>
<tr>
<td>↓ $\pi_4$</td>
</tr>
<tr>
<td>30154</td>
</tr>
<tr>
<td>↓ $\pi_3$</td>
</tr>
<tr>
<td>30514</td>
</tr>
<tr>
<td>↓ $\pi_2$</td>
</tr>
<tr>
<td>35014</td>
</tr>
<tr>
<td>↓ $\pi_1$</td>
</tr>
<tr>
<td>53014</td>
</tr>
<tr>
<td>↓ $\pi_0$</td>
</tr>
<tr>
<td>03014</td>
</tr>
<tr>
<td>↓ $\pi_1$</td>
</tr>
<tr>
<td>30014</td>
</tr>
</tbody>
</table>
Canonical reduced expression

Example: using coset $R^0_5 / R^0_4$

30145
↓ $\pi_4$
30154
↓ $\pi_3$
30514
↓ $\pi_2$
35014
↓ $\pi_1$
53014
↓ $\pi_0$
03014
↓ $\pi_1$
30014
$\circlearrowright$ $\pi_2$
Canonical reduced expression

Example: 30240

\[
\begin{align*}
\pi_0 & \\
\pi_1 & \\
\pi_2 & \\
\pi_3 & \\
\pi_4 & \\
\end{align*}
\]

Conclusion:

\[
\begin{align*}
\pi_0 & \\
\pi_1 & \\
\pi_2 & \\
\pi_3 & \\
\pi_4 & \\
\end{align*}
\] = 30240.
Canonical reduced expression

Example: 30240

Index the zeros by the missing letters in decreasing order: 30_524_0_1

12345  \text{ 1}_5
Canonical reduced expression

Example: 30240

Index the zeros by the missing letters in decreasing order: 3052401

```
12345
012345
· π0
15
π0
```

Conclusion: $[\pi_0 \cdot \pi_1 \cdot \pi_2 \pi_1 \cdot \pi_3 \cdot \pi_4 \pi_3 \pi_2 \pi_1 \pi_0 \pi_1] = 30240$. 
Canonical reduced expression

Example: 30240

Index the zeros by the missing letters in decreasing order: 30_240_1

\[
\begin{align*}
&12345 & 1_5 \\
&0_12345 & \cdot \pi_0 \\
&20_1345 & \cdot \pi_1
\end{align*}
\]
Canonical reduced expression

Example: 30240

Index the zeros by the missing letters in decreasing order: 30⁵₂₄₀₁

| 12345 | 1₅ |
|       |     |
| 0₁₂₃₄₅ | ⋅  π₀ |
| 2₀₁₃₄₅ | ⋅  π₁ |
| 3₂₀₁₄₅ | ⋅  π₂π₁ |
Canonical reduced expression

Example: 30240

Index the zeros by the missing letters in decreasing order: 30_5240_1

\[
\begin{array}{l}
12345 \\
0_12345 \\
20_1345 \\
320_145 \\
3240_15 \\
1_5 \\
\cdot \pi_0 \\
\cdot \pi_1 \\
\cdot \pi_2 \pi_1 \\
\cdot \pi_3
\end{array}
\]
## Canonical reduced expression

**Example: 30240**

Index the zeros by the missing letters in decreasing order: $30_5240_1$

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>12345</td>
<td>$\pi_0$</td>
</tr>
<tr>
<td>0_12345</td>
<td>$\pi_1$</td>
</tr>
<tr>
<td>20_1345</td>
<td>$\pi_2\pi_1$</td>
</tr>
<tr>
<td>320_145</td>
<td>$\pi_3$</td>
</tr>
<tr>
<td>3240_15</td>
<td>$\pi_4\pi_3\pi_2\pi_1\pi_0\pi_1$</td>
</tr>
</tbody>
</table>

Conclusion: $\pi_0\pi_1\pi_2\pi_3\pi_4 = 30_5240_1$.  

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### Example: 30240

Index the zeros by the missing letters in decreasing order: $30_5240_1$

<table>
<thead>
<tr>
<th>12345</th>
<th>$1_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>012345</td>
<td>$\pi_0$</td>
</tr>
<tr>
<td>201345</td>
<td>$\pi_1$</td>
</tr>
<tr>
<td>320145</td>
<td>$\pi_2 \pi_1$</td>
</tr>
<tr>
<td>324015</td>
<td>$\pi_3$</td>
</tr>
<tr>
<td>3052401</td>
<td>$\pi_4 \pi_3 \pi_2 \pi_1 \pi_0 \pi_1$</td>
</tr>
</tbody>
</table>

Conclusion: $1_5 \cdot [\pi_0 \cdot \pi_1 \cdot \pi_2 \pi_1 \cdot \pi_3 \cdot \pi_4 \pi_3 \pi_2 \pi_1 \pi_0 \pi_1] = 30240$. 
\(\mathcal{J}\)-triviality

**Definition (Green)**

Let \( M \) a monoid, \( x, y \in M \). We say that \( x \preceq_{\mathcal{J}} y \) iff \( MxM \subseteq MyM \).

Equivalence relation : \( x\mathcal{J}y \) iff \( MxM = MyM \).

**Definition**

A monoid is \(\mathcal{J}\)-trivial if its \(\mathcal{J}\)-classes are trivial. Equivalently, its bisided Cayley graph has no cycle except loops.

Example : \( H^0_n \)
$J$-triviality: $H_n^0$ right Cayley graph
$\mathcal{J}$-triviality: $H^0_n$ left Cayley graph
$\mathcal{J}$-triviality: $H^0_n$ bisided Cayley graph
$\mathcal{J}$-triviality : $R^0_n$ bisided Cayley graph

Joël Gay

The 0-Rook Monoid
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4 Work in Progress
Simple modules

Theorem

\( R_n^0 \) is \( \mathcal{J} \)-trivial.
Simple modules

Theorem

$R_n^0$ is $\mathcal{J}$-trivial.

Corollary (Application of Denton-Hivert-Schilling-Thiery)

$R_n^0$ has $2^n$ idempotents.

It has thus $2^n$ simple modules of dimension 1.
Descent set

**Definition**

For $\pi \in R_n^0$, we define its right $R$-descent set by

$$D_R(\pi) = \{0 \leq i \leq n-1 \mid \pi \pi_i = \pi\}.$$
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Example: positions

Let $r = 0423007$. $0 < 4 \geq 2 < 3 \geq 0 \geq 0 < 7$.

$D_R(r) = \{0, 2, 4, 5\}$
Descent set

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**Example: positions**

Let $r = 0423007$. $0 < 4 \geq 2 < 3 \geq 0 \geq 0 < 7$.

$$D_R(r) = \{0, 2, 4, 5\} \quad \text{Notation:} \quad \begin{array}{c}
0 & 4 \\
2 & 3 \\
0 & 7
\end{array}$$

*Warning:* Not $0$. 
Descent set

**Definition**

For \( \pi \in R_n^0 \), we define its right \( R \)-descent set by

\[
D_R(\pi) = \{ 0 \leq i \leq n - 1 \mid \pi \pi_i = \pi \}.
\]

**Example: positions**

Let \( r = 0423007 \). \( 0 < 4 \geq 2 < 3 \geq 0 \geq 0 < 7 \).

\[
D_R(r) = \{0, 2, 4, 5\} \quad \text{Notation:}
\]

\[
\begin{array}{c}
0 \\
2 \\
0 \\
0
\end{array}
\begin{array}{c}
4 \\
3 \\
7 \\
0
\end{array}
\]

Warning: \( 0 \) and not \( 00 \).
List of the $R$-descent types for $R^0_4$: 

\[
\begin{array}{cccc}
\{\} & \{0\} & \{1\} & \{2\} \\
\{1, 2\} & \{1, 3\} & \{2, 3\} & \{0, 1, 2\} \\
\{0, 1, 3\} & \{0, 2, 3\} & \{1, 2, 3\} & \{0, 1, 2, 3\}
\end{array}
\]
The projective indecomposable $R^0_n$-modules are indexed by the $R$-descent type and isomorphic to the quotient of the associated $R$-descent class by the finer $R$-descent class.
Projectivity over $H_n^0$

**Theorem**

The indecomposable projective $R_n^0$-module splits as a $H_n^0$-module as the direct sum of all the indecomposable projective $H_n^0$-modules whose descent classes are explicit.

**Proof**: explicit decomposition

\[
\begin{align*}
0 &= 0 + 0 = \begin{array}{c}
\text{yellow}
\end{array} + \begin{array}{c}
\text{blue}
\end{array} = \begin{array}{c}
\text{yellow}
\end{array} + \begin{array}{c}
\text{blue}
\end{array} + \begin{array}{c}
\text{yellow}
\end{array} + \begin{array}{c}
\text{blue}
\end{array} + \begin{array}{c}
\text{yellow}
\end{array} + \begin{array}{c}
\text{blue}
\end{array} + \begin{array}{c}
\text{yellow}
\end{array} + \begin{array}{c}
\text{blue}
\end{array} + 2 \begin{array}{c}
\text{yellow}
\end{array} + \begin{array}{c}
\text{blue}
\end{array}.
\end{align*}
\]
Symmetric Group and Rook Monoid

The \(0\)-Rook Monoid

Representation theory

Work in Progress

\[
\begin{array}{c}
0 = \begin{array}{c}
0
\end{array} + \begin{array}{c}
0
\end{array} = \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
0
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
0
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\end{array}.
\end{array}
\]
$$0 = 0 + 0 + 0 = 0 + 0 + 0 + 0 + 0 + 0 + 0 + 2 + 0.$$
\[ 0 = 0 + 0 + 0 + 0 = 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0 + 0 + 0 + 0 + 0 + 0 + 0 + 2 + 0. \]
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4 Work in Progress
\( R_n^0 \) is a lattice (analogous to permutohedron)
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- Tower of monoids: induction and restriction (linked to QSym and NCSF, work of Krob and Thibon)
• $R^0_n$ is a lattice (analogous to permutohedron)
• Tower of monoids: induction and restriction (linked to QSym and NCSF, work of Krob and Thibon)
• Renner Monoids (generalization for other Cartan types)
THANK YOU FOR YOUR OUTSTANDING ATTENTION!!
Descent classes are not intervals