The 0-Rook Monoid

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Contents

$$\mathfrak{S}_n$$

Symmetric Group

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q)$$

Symmetric Group

Iwahori-Hecke algebra

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q) \xrightarrow{q=0} H_n^0$$

Symmetric Group

Iwahori-Hecke algebra

Hecke monoid at $q = 0$

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Symmetric Group	Iwahori-Hecke algebra	Hecke monoid at $q = 0$
$s_i^2 = 1$	$T_i^2 = q1 + (q-1)T_i$	$\pi_i^2 = \pi_i$
$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$	$T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$	$\pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i$
$s_is_j = s_js_i$	$T_iT_j = T_jT_i$	$\pi_i\pi_j = \pi_j\pi_i$

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$s_is_j = s_js_i$	$T_iT_j = T_jT_i$	$\pi_i\pi_j = \pi_j\pi_i$

$s_i = T_i$ $\qquad$ $\pi_i = T_i + 1$

Interesting properties of H_n^0

- $|H_n^0| = n!$

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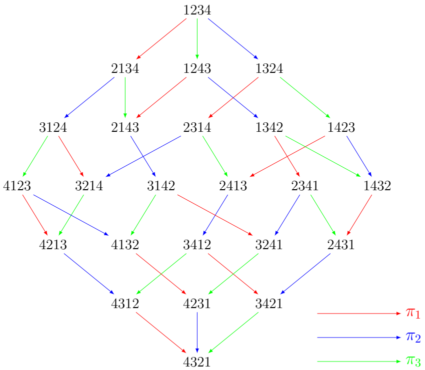
- An element of H_n^0 is characterized by its action on the identity.



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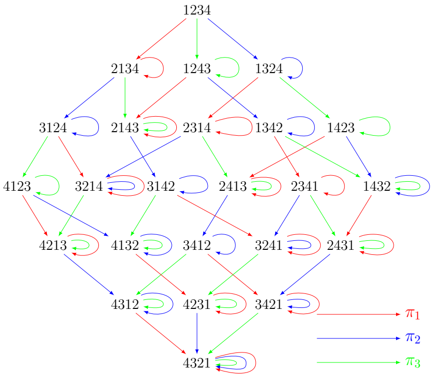

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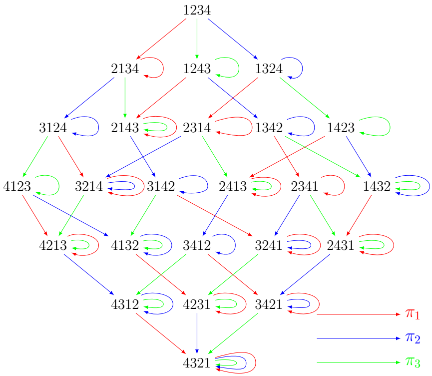

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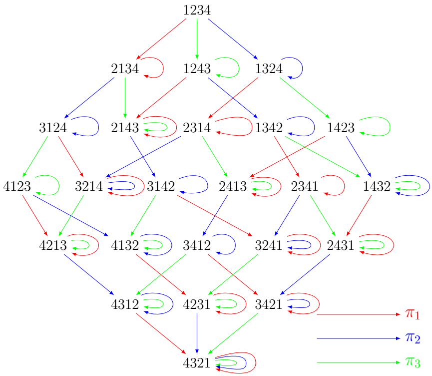
- An element of H_n^0 is characterized by its action on the identity.
- Its simple and projective modules are well-known and combinatorial [Norton-Carter].
- The induction and restriction of modules gives us a structure of tower of monoids, linked to QSym and NCSF [Krob-Thibon].



The rook monoid

Rook matrix of size n = set of non attacking rooks on an $n \times n$ matrix.

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$$\text{Rook Matrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & \text{♜} & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\text{Rook Vector} \quad \begin{matrix} 0 & 4 & 2 & 3 & 1 \end{matrix} \qquad \begin{matrix} 0 & 3 & 0 & 4 & 1 \end{matrix}$$

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$$
\begin{array}{lcc}
\text{Rook Matrix} &
\begin{pmatrix}
0 & 0 & 0 & 0 & \text{♜} \\
0 & 0 & \text{♜} & 0 & 0 \\
0 & 0 & 0 & \text{♜} & 0 \\
0 & \text{♜} & 0 & 0 & 0 \\
\text{👻} & 0 & 0 & 0 & 0
\end{pmatrix}
&
\begin{pmatrix}
0 & 0 & 0 & 0 & \text{♜} \\
0 & 0 & \text{👻} & 0 & 0 \\
0 & \text{♜} & 0 & 0 & 0 \\
0 & 0 & 0 & \text{♜} & 0 \\
\text{👻} & 0 & 0 & 0 & 0
\end{pmatrix}
\\
\text{Rook Vector} & 0_5 \quad 4 \quad 2 \quad 3 \quad 1 & 0_5 \quad 3 \quad 0_2 \quad 4 \quad 1
\end{array}
$$

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$$\text{Rook Vector} \quad 0_5 \; 4 \; 2 \; 3 \; 1 \qquad 0_5 \; 3 \; 0_2 \; 4 \; 1$$

The product of two rook matrices is a rook matrix.
Rook Monoid R_n = submonoid of the rook matrices
$M_n \supset R_n \supset \mathfrak{S}_n$

$\mathfrak{S}_n$

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q)$$

Iwahori

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$$\downarrow$$

$$R_n$$

Iwahori

$$\begin{array}{ccccc} \mathfrak{S}_n & \xleftarrow{q=1} & \mathcal{H}_n(q) & \xrightarrow{q=0} & H_n^0 \\ \downarrow & & \downarrow & & \\ R_n & \xleftarrow{q=1} & \mathcal{I}_n(q) & & \end{array}$$

Iwahori , Solomon.

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Iwahori , Solomon.

Contents

Definition by right action on R_n

Operators $\pi_0, \pi_1, \dots \pi_{n-1}$ acting on rook vectors

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Bubble sort operators $\pi_1, \dots, \pi_{n-1}$:

$$(r_1 \dots r_n) \cdot \pi_i = \begin{cases} r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n & \text{if } r_i < r_{i+1}, \\ r_1 \dots r_n & \text{otherwise,} \end{cases}$$

Deletion operator π_0 :

$$(r_1 \dots r_n) \cdot \pi_0 = \quad 0 r_2 \dots r_n.$$

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$$45321 \cdot \pi_1 = 54321$$
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$$00321 \cdot \pi_2 = 03021$$
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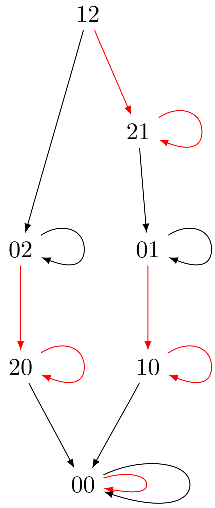

Definition by presentation

Generators : $\pi_0, \ldots, \pi_{n-1}$

Relations :

$$\begin{aligned} \pi_i^2 &= \pi_i & 0 \leq i \leq n-1, \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} & 1 \leq i \leq n-2, \\ \pi_i \pi_j &= \pi_j \pi_i & |i-j| > 1. \\ \pi_1 \pi_0 \pi_1 \pi_0 &= \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 \pi_1 \end{aligned}$$

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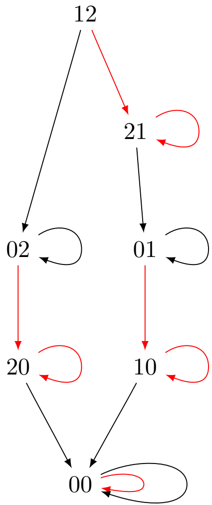
\end{aligned}
$$



Theorem

Both definitions (presentation and action on R_n) are equivalent.

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Key Fact :

Theorem

The map $f : \left| \begin{array}{ccc} R_n^0 & \longrightarrow & R_n \\ r & \longmapsto & \mathbf{1}_n \cdot r \end{array} \right.$ *is a bijection.*

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The map $f : \left| \begin{array}{ccc} R_n^0 & \longrightarrow & R_n \\ r & \longmapsto & \mathbf{1}_n \cdot r \end{array} \right.$ *is a bijection.*

This also gives us canonical *reduced expression* of elements of R_n^0.

Canonical reduced expression

Example : using coset R_5^0 / R_4^0

30145

Canonical reduced expression

Example : using coset R_5^0/R_4^0

$$30145$$

$$\downarrow \pi_4$$

$$30154$$

Canonical reduced expression

Example : using coset R_5^0 / R_4^0

$$30145$$
$$\downarrow \pi_4$$
$$30154$$
$$\downarrow \pi_3$$
$$30514$$

Canonical reduced expression

Example : using coset R_5^0 / R_4^0

$$
\begin{array}{c}
3014\color{red}{5} \\
\downarrow \pi_4 \\
301\color{red}{5}4 \\
\downarrow \pi_3 \\
30\color{red}{5}14 \\
\downarrow \pi_2 \\
3\color{red}{5}014
\end{array}
$$

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\downarrow \pi_2 \\
3\color{red}{5}014 \\
\downarrow \pi_1 \\
\color{red}{5}3014
\end{array}
$$

Canonical reduced expression

Example : using coset R_5^0 / R_4^0

$$30145 \;\downarrow \pi_4\; 30154 \;\downarrow \pi_3\; 30514 \;\downarrow \pi_2\; 35014 \;\downarrow \pi_1\; 53014 \;\downarrow \pi_0\; 03014$$

Canonical reduced expression

Example : using coset R_5^0 / R_4^0

$$30145$$
$$\downarrow \pi_4$$
$$30154$$
$$\downarrow \pi_3$$
$$30514$$
$$\downarrow \pi_2$$
$$35014$$
$$\downarrow \pi_1$$
$$53014$$
$$\downarrow \pi_0$$
$$03014$$
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$$30014$$

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3\color{red}{5}014 \\
\downarrow \pi_1 \\
\color{red}{5}3014 \\
\downarrow \pi_0 \\
\color{red}{0}3014 \\
\downarrow \pi_1 \\
3\color{red}{0}014 \\
\circlearrowleft \pi_2
\end{array}
$$

Canonical reduced expression

Example : 30240

Canonical reduced expression

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Index the zeros by the missing letters in decreasing order : 30_5240_1

12345 $\mathbf{1}_5$

Canonical reduced expression

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$$\begin{array}{cc} 12345 & \mathbf{1}_5 \\ 0_1 2345 & \cdot\, \pi_0 \end{array}$$

Canonical reduced expression

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$2 0_1 345$	$\cdot\, \pi_1$

Canonical reduced expression

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$20_1 345$	$\cdot\, \pi_1$
$320_1 45$	$\cdot\, \pi_2 \pi_1$

Canonical reduced expression

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

$$\begin{array}{lr} 12345 & \mathbf{1}_5 \\ 0_1 2345 & \cdot\, \pi_0 \\ 20_1 345 & \cdot\, \pi_1 \\ 320_1 45 & \cdot\, \pi_2\pi_1 \\ 3240_1 5 & \cdot\, \pi_3 \end{array}$$

Canonical reduced expression

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12345	$\mathbf{1}_5$
$0_1 2345$	$\cdot\, \pi_0$
$20_1 345$	$\cdot\, \pi_1$
$320_1 45$	$\cdot\, \pi_2\pi_1$
$3240_1 5$	$\cdot\, \pi_3$
$30_5 240_1$	$\cdot\, \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1$

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$30_5 240_1$	$\cdot \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1$

Conclusion : $\mathbf{1}_5 \cdot [\pi_0 \cdot \pi_1 \cdot \pi_2\pi_1 \cdot \pi_3 \cdot \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1] = 30240.$

$\mathcal{J}$-triviality

Definition (Green)

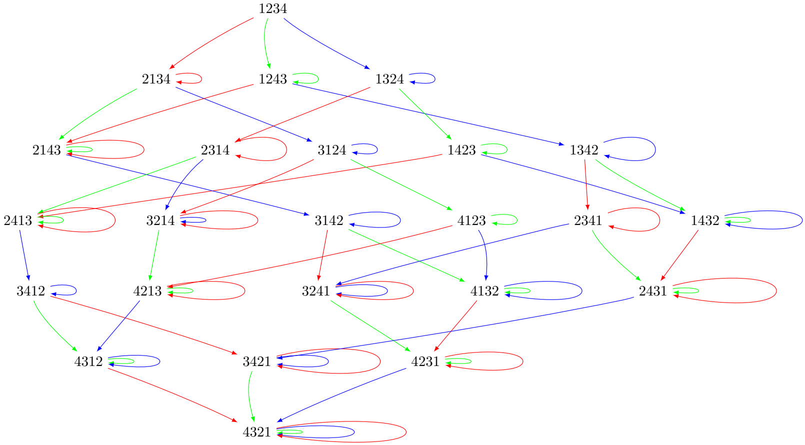
Let M a monoid, $x, y \in M$. We say that $x \leq_{\mathcal{J}} y$ iff $MxM \subseteq MyM$.
Equivalence relation : $x\mathcal{J}y$ iff $MxM = MyM$.

Definition

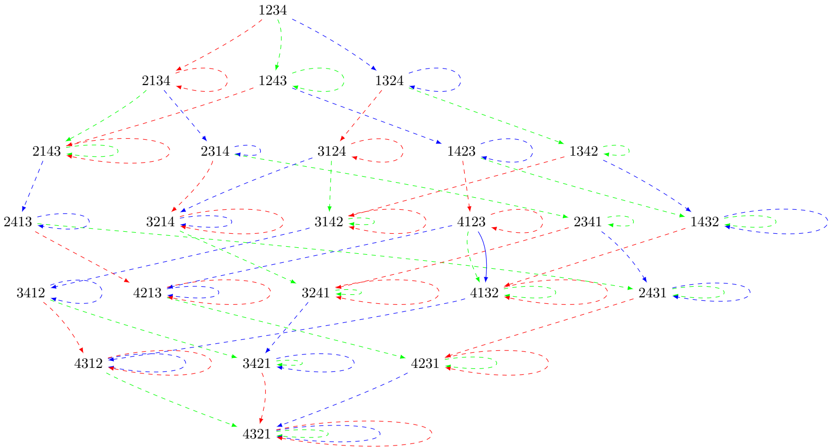
A monoid is $\mathcal{J}$-trivial if its $\mathcal{J}$-classes are trivial.
Equivalently, its bisided Cayley graph has no cycle except loops.

Example : H_n^0

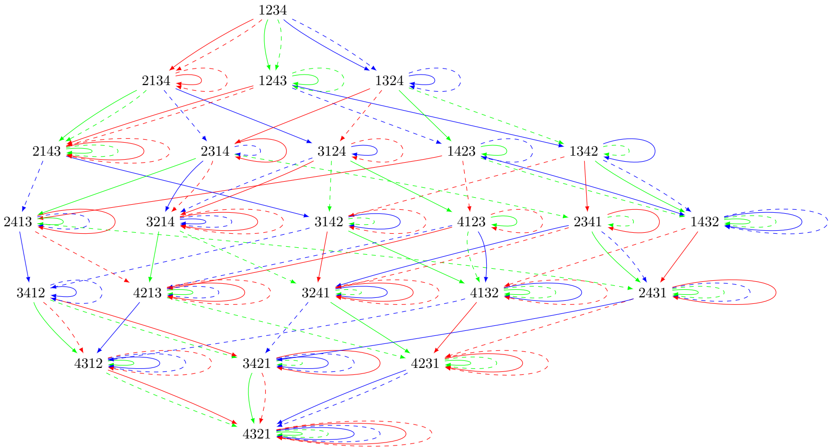
$\mathcal{J}$-triviality : H_n^0 right Cayley graph



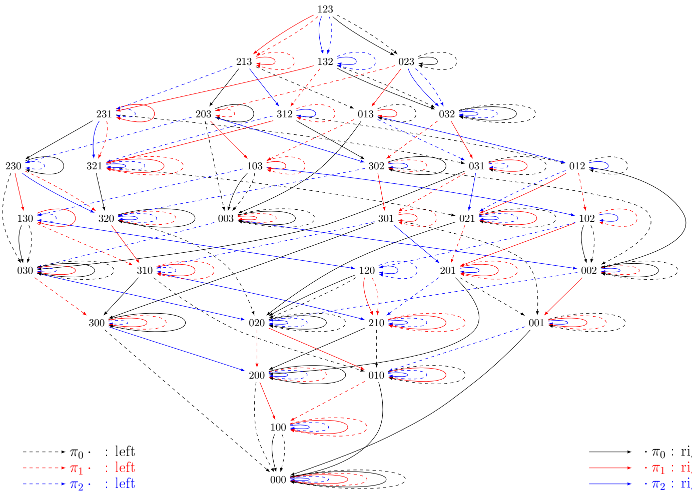
$\mathcal{J}$-triviality : H_n^0 left Cayley graph



$\mathcal{J}$-triviality : H_n^0 bisided Cayley graph



$\mathcal{J}$-triviality : R_n^0 bisided Cayley graph



Contents

Simple modules

Theorem

R_n^0 is $\mathcal{J}$-trivial.

Simple modules

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R_n^0 is $\mathcal{J}$-trivial.

Corollary (Application of Denton-Hivert-Schilling-Thiéry)

R_n^0 has 2^n idempotents.
It has thus 2^n simple modules of dimension 1.

Descent set

Definition

For $\pi \in R_n^0$, we define its right R-descent set by

$$D_R(\pi) = \{0 \leq i \leq n-1 \mid \pi\pi_i = \pi\}.$$

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Example : positions

Let $r = 0423007$. $0 < 4 \ge 2 < 3 \ge 0 \ge 0 < 7$.

$D_R(r) = \{0, 2, 4, 5\}$

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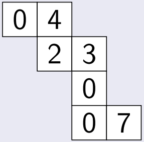

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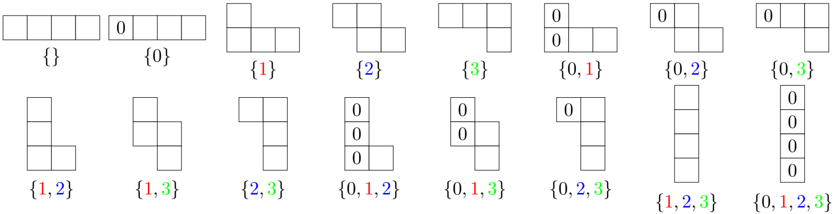
$D_R(r) = \{0, 2, 4, 5\}$ Notation :

$$\begin{array}{|c|c|}\hline 0 & 4 \\ \hline \end{array}$$
$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}$$
$$\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline \end{array}\begin{array}{c|c|}\\ \hline 7 \\ \hline \end{array}$$

Warning : $\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline \end{array}$ and not $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline \end{array}$.

Descent class

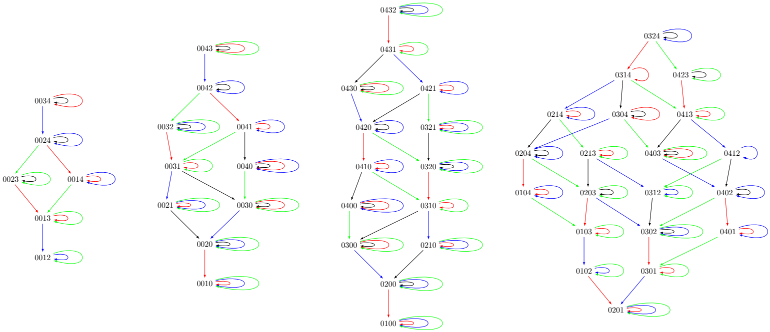
List of the *R*-descent types for R_4^0:



Projective modules

Theorem (Application of Denton-Hivert-Schilling-Thiéry)

The projective indecomposable R_n^0-modules are indexed by the R-descent type and isomorphic to the quotient of the associated R-descent class by the finer R-descent class.

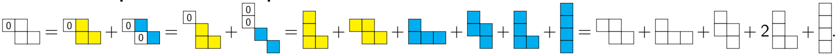


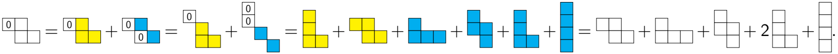
Projectivity over H_n^0

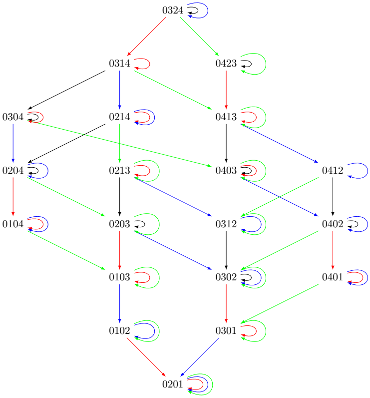
Theorem

The indecomposable projective R_n^0-module splits as a H_n^0-module as the direct sum of all the indecomposable projective H_n^0-modules whose descent classes are explicit.

Proof : explicit decomposition



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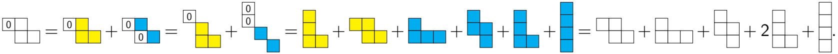

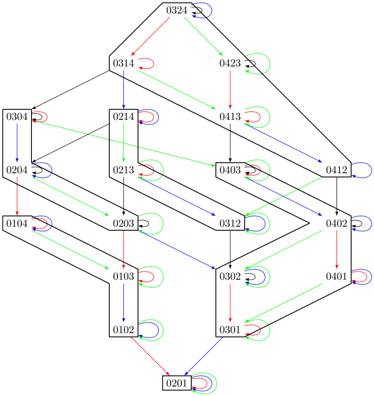
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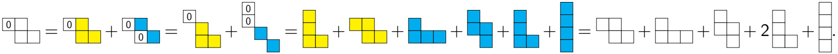

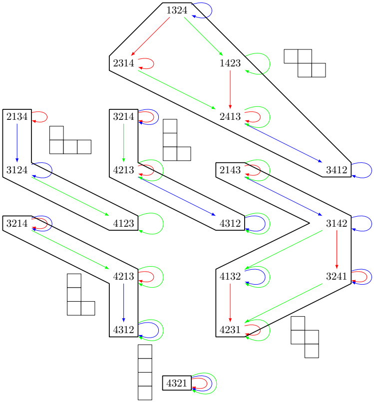
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Contents

- R_n^0 is a lattice (analogous to permutohedron)

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- Tower of monoids : induction and restriction (linked to QSym and NCSF, work of Krob and Thibon)
- Renner Monoids (generalization for other Cartan types)

THANK YOU FOR YOUR OUTSTANDING ATTENTION!!

Descent classes are not intervals