On (rational) Shi tableaux

Robin Sulzgruber

78th Séminaire Lotharingien de Combinatoire
March 26th–29th 2017 • Ottrott • France
Setting the stage

Definition Let \( V \) be a Euclidean vector space, \( \alpha \in V \) a non-zero vector and \( k \in \mathbb{Z} \). Define the affine hyperplane \( H_{\alpha, k} = \{ x \in V : \langle x, \alpha \rangle = k \} \).

Define the reflection in \( H_{\alpha, k} \) as \( s_{\alpha, k}(x) = x + 2k - \frac{\langle x, \alpha \rangle \langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \).

Definition An irreducible crystallographic root system is a finite subset \( \Phi \subseteq V \) with some properties.

The Weyl group of \( \Phi \) is the group generated by the reflections \( s_{\alpha, 0} \) for \( \alpha \in \Phi^+ \).
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The root system of type $A_2$
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The root system of type $A_2$

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$H_{\alpha_1,0}$

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$H_{\alpha_2,0}$

$\alpha_1 = e_1 - e_2$

$H_{\alpha_1+\alpha_2,0}$
The root system of type $A_2$

The reflections $s_{\alpha_1,0}, s_{\alpha_2,0}$ generate the symmetric group $\mathfrak{S}_3$. 
The affine Weyl group

Definition The affine arrangement of $\Phi$ is defined as

$$\text{Aff} = \{ H_\alpha, k : \alpha \in \Phi^+, k \in \mathbb{Z} \}.$$ 

The regions of the affine arrangement are called alcoves.

The affine Weyl group $\tilde{W}$ is the group generated by all reflections in the hyperplanes of $\text{Aff}$. It acts simply transitively on the set of alcoves.
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The Shi arrangement

Definition The Shi arrangement is defined as

\[ \text{Shi} = \{ \alpha, k : \alpha \in \Phi^+, k \in \{0, 1\} \} \].

Theorem (Shi 1987, 1997) The Shi arrangement has \((h + 1)r\) regions and

1 \mid W \prod_{i=1}^{\text{dominant regions}} (d_i + h).
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\[ (n + 1)^{n-1} \]
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\frac{1}{n+1} \binom{2n}{n}
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\[
\frac{(n + 1)^{n-1}}{n+1} = 4^2 = 16
\]

\[
\frac{1}{n+1} \binom{2n}{n} = \frac{6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 5
\]
The $m$-Shi arrangement

Definition The $m$-Shi arrangement is defined as

$$\text{Shi}_m = \{ H_\alpha, k : \alpha \in \Phi^+, -m < k \leq m \}.$$

Theorem (Athanasiadis 2004, Yoshinaga 2004) The Shi arrangement has

$$\left(mh + 1\right)$$

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$$\frac{1}{|W|} \prod_{i=1}^{r} (d_i + mh) = (mn + 1)^{n-1}$$
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$$(mn + 1)^{n-1} = 49$$
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\frac{1}{mn + 1} \binom{mn + n}{n}
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$$\left(\frac{mn + 1}{mn + 1} \binom{mn + n}{n}\right) = 12$$

$$\left(\frac{mn + 1}{mn + 1} \binom{mn + n}{n}\right) = 49$$
Walls and floors

Definition A hyperplane $H_\alpha, k$ is called wall of an alcove if it supports a facet of the alcove. A wall is called floor if it separates the alcove from the fundamental alcove.

$[4, 2, 0]$
Walls and floors

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The height of a hyperplane

Definition

Define the height of a hyperplane $H_{\alpha,k}$ as $|ht(\alpha) - h_k|$.
**The height of a hyperplane**

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Definition Define the height of a hyperplane $H_{\alpha,k}$ as $|\text{ht}(\alpha) - hk|$.
Theorem (Shi 1987, Athanasiadis 2005, Thiel 2015) The regions of the $m$-Shi arrangement are in bijection with alcoves whose floors have height less than $m_h + 1$. 

\[
\begin{align*}
&[4, 2, 0] \\
&[1, -1, 6] \\
&[2, 0, 4] \\
&[1, 0, 5] \\
&[1, 2, 3] \\
&[2, 1, 3] \\
&[2, 3, 1] \\
&[3, 1, 2] \\
&[1, 3, 2] \\
&[3, ..., 5, 3] \\
&[−1, 4, 3] \\
&[0, 4, 2] \\
\end{align*}
\]
Theorem (Shi 1987, Athanasiadis 2005, Thiel 2015) The regions of the \( m \)-Shi arrangement are in bijection with alcoves whose floors have height less than \( mh + 1 \).
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Diagram showing regions of the $m$-Shi arrangement with specific alcoves labeled.
Theorem (Fishel, Vazirani 2010) The regions of the $m$-Shi arrangement are in bijection with the alcoves inside the simplex bounded by the hyperplanes of height $mh + 1$. 

$\begin{align*}
-2, 2, 6 \\
1, 5, 0 \\
0, 1, 5 \\
1, 0, 5 \\
1, 2, 3 \\
2, 1, 3 \\
3, 1, 2 \\
2, 3, 1 \\
1, 3, 2 \\
-1, 3, 4 \\
-1, 4, 3 \\
-1, 2, 4 \\
0, 4, 2 \\
2, 0, 4 \\
0, 2, 4 \\
16, 13, 10, 7, 1, 2, 5 \\
11 \\
14 \\
14, 11, 8, 5, 2, 1, 7 \\
10 \\
13 \\
16 \\
7 \\
1 \\
2 \\
5 \\
8 \\
11 \\
4 \\
4 \\
4
\end{align*}$
Inverse Shi alcoves

Theorem (Fishel, Vazirani 2010) The regions of the $m$-Shi arrangement are in bijection with the alcoves inside the simplex bounded by the hyperplanes of height $mh + 1$. 
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A rational analogue

Definition Let $p$ be a positive integer relatively prime to the Coxeter number $h$. An alcove is called $p$-stable if its inverse lies inside the simplex bounded by the hyperplanes of height $p$.

Theorem (Thiel 2015) The number of $p$-stable alcoves equals $p^r$. The number of dominant $p$-stable alcoves equals \[ \frac{1}{|W|} r \prod_{i=1}^{\infty} \left( p + e_i \right). \]
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Shi tableaux

Definition (Fishel, Tzanaki, Vazirani 2011) Let $w(\circ)$ be a dominant Shi alcove and $\alpha \in \Phi^+$. Define $t_{mh+1}(\alpha, w)$ as the number of Shi hyperplanes $H_{\alpha, k}$ that separate $w(\circ)$ and $\circ$.

The Shi tableau of $w$ is the collection of the numbers $t_{mh+1}(\alpha, w)$ for $\alpha \in \Phi^+$.
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$w = [4, 2, 0]$

$t^4(\alpha_1, w) = 1$

$t^4(\alpha_2, w) = 1$

$t^4(\alpha_1 + \alpha_2, w) = 1$
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\[
\begin{array}{ccc}
4 & 2 & 0 \\
2 & 0 & 4 \\
1 & 2 & 3 \\
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Definition (Fishel, Tzanaki, Vazirani 2011) Let $w(A_\circ)$ be a dominant Shi alcove and $\alpha \in \Phi^+$. Define $t_{mh+1}^{\alpha}(w)$ as the number of Shi hyperplanes of the form $H_{\alpha,k}$ that separate $w(A_\circ)$ and $A_\circ$.

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Definition Let $w(A^\circ)$ be dominant and $p$-stable and $\alpha \in \Phi^+$. Define $t_p(\alpha, w)$ as the number of hyperplanes of the form $H_{\alpha, k}$ with height less than $p$ that separate $w(A^\circ)$ and $A^\circ$. The rational Shi tableau of $w$ is defined as the collection of numbers $t_p(\alpha, w)$ for $\alpha \in \Phi^+$. 

\[
\begin{bmatrix}
-3 & 2 & 7 \\
2 & 0 & 4 \\
0 & 2 & 4 \\
4 & 0 & 2 \\
1 & 2 & 3 \\
0 & 4 & 2 \\
2 & 4 & 0
\end{bmatrix}
\]
Rational Shi tableaux

Definition Let \( w(A_\circ) \) be dominant and \( p \)-stable and \( \alpha \in \Phi^+ \). Define \( t^p(\alpha, w) \) as the number of hyperplanes of the form \( H_{\alpha,k} \) with height less than \( p \) that separate \( w(A_\circ) \) and \( A_\circ \).
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- $t_5(\alpha_1, w) = 1$
- $t_5(\alpha_2, w) = 1$
- $t_5(\alpha_1 + \alpha_2, w) = 2$
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\[
\begin{align*}
  w &= [-3, 2, 7] \\
  t^5(\alpha_1, w) &= 1 \\
  t^5(\alpha_2, w) &= 1 \\
  t^5(\alpha_1 + \alpha_2, w) &= 2
\end{align*}
\]
The Main Conjecture

Conjecture Every dominant $p$-stable element $w \in \tilde{W}$ is uniquely determined by its rational Shi tableau.

Theorem The conjecture is true in type $A^{n-1}$.

Open Problem Characterise the set of rational Shi tableaux.
The Main Conjecture

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**Theorem** The conjecture is true in type $A_{n-1}$.
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**Theorem** The conjecture is true in type $A_{n-1}$.

**Open Problem** Characterise the set of rational Shi tableaux.
Inverting the rational Shi tableau in type $A_{n-1}$

Example Consider the affine permutation of type $A_4$

$w = [7, -1, 11, 3, -5]$. Then the alcove of $w^{-1}$ is contained in the simplex bounded by the hyperplanes of height $p = 8$. The Shi tableau of $w$ is given by:

$\begin{array}{cccc}
0 & 2 & 1 & 2 \\
1 & 2 & 1 & 0 \\
2 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
\end{array}$
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Then the alcove of $w^{-1}$ is contained in the simplex bounded by the hyperplanes of height $p = 8$.

The Shi tableau of $w$ is given by

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\begin{align*}
\alpha_{15} & \quad 2 \quad \alpha_{25} & \quad 1 \quad \alpha_{35} & \quad 2 \quad \alpha_{45} & \quad 1 \\
\alpha_{14} & \quad 1 \quad \alpha_{24} & \quad 2 \quad \alpha_{34} & \quad 0 \\
\alpha_{13} & \quad 2 \quad \alpha_{23} & \quad 1 \\
\alpha_{12} & \quad 0
\end{align*}
\]
To Dyck paths via row-sums and column-sums

$$\begin{align*}
\alpha_{15} & 2 \quad \alpha_{25} & 1 \quad \alpha_{35} & 2 \quad \alpha_{45} & 1 \\
\alpha_{14} & 1 \quad \alpha_{24} & 2 \quad \alpha_{34} & 0 \\
\alpha_{13} & 2 \quad \alpha_{23} & 1 \\
\alpha_{12} & 0
\end{align*}$$
To Dyck paths via row-sums and column-sums

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\[ \alpha_{14} \ 1 \ \alpha_{24} \ 2 \ \alpha_{34} \ 0 \]

\[ \alpha_{13} \ 2 \ \alpha_{23} \ 1 \]

\[ \alpha_{12} \ 0 \]
To long cycles (Ceballos, Denton, Hanusa 2016)
To long cycles \((\text{Ceballos, Denton, Hanusa 2016})\)
To long cycles (Ceballos, Denton, Hanusa 2016)

\[(4, 2, 6, 9, 7, 11, 13, 12, 10, 8, 5, 3, 1)\]
(4, 2, 6, 9, 7, 11, 13, 12, 10, 8, 5, 3, 1)
Back to Dyck paths (Ceballos, Denton, Hanusa 2016)

\((4, 2, 6, 9, 7, 11, 13, 12, 10, 8, 5, 3, 1)\)
Back to Dyck paths (Ceballos, Denton, Hanusa 2016)

(4, 2, 6, 9, 7, 11, 13, 12, 10, 8, 5, 3, 1)
To \( n \) and \( p \) flush abaci (Anderson 2002)
To $n$ and $p$ flush abaci (Anderson 2002)
To \( n \) and \( p \) flush abaci (Anderson 2002)

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To $n$ and $p$ flush abaci (Anderson 2002)
To $n$ and $p$ flush abaci (Anderson 2002)
Shift back to affine permutations (Lascoux 2001)
Shift back to affine permutations (Lascoux 2001)

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-14 & -13 & -12 & -11 & -10 & \\
-9 & -8 & -7 & -6 & -5 & \\
-4 & -3 & -2 & -1 & 0 & \\
1 & 2 & 3 & 4 & 5 & \\
6 & 7 & 8 & 9 & 10 & \\
11 & 12 & 13 & 14 & 15 & \\
16 & 17 & 18 & 19 & 20 & \\
21 & 22 & 23 & 24 & 25 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
Shift back to affine permutations ([Lascoux 2001])

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{-14} & \text{-13} & \text{-12} & \text{-11} & \text{-10} & \\
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\text{-4} & \text{-3} & \text{-2} & \text{-1} & \text{0} & \\
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\text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \\
\text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \\
\text{16} & \text{17} & \text{18} & \text{19} & \text{20} & \\
\text{21} & \text{22} & \text{23} & \text{24} & \text{25} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

\[
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Shift back to affine permutations (Lascoux 2001)

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\quad
\begin{array}{ccccccc}
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11 & 12 & 13 & 14 & 15 & \cdot & \cdot \\
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\end{array}
\]

\[w^{-1} = [-7, -4, 4, 7, 15]\]
Shift back to affine permutations (Lascoux 2001)

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$w^{-1} = [-7, -4, 4, 7, 15]$  
$w = [7, -1, 11, 3, -5]$
This is the end.

Thank you!
Shi coordinates
Shi coordinates
Sign types

[Diagram showing sign types with labels such as $H_{13,1}$ and $H_{12,0}$]
On (rational) Shi tableaux

### Table of Contents

1. Introduction
2. Basic Concepts
3. Rational Shi Tableaux
4. Applications
5. Conclusion

### Key Points

- **Introduction**
  - Overview of Shi tableaux
  - Importance in combinatorics

- **Basic Concepts**
  - Definition of Shi tableaux
  - Properties and applications

- **Rational Shi Tableaux**
  - Definition and examples
  - Properties and applications

- **Applications**
  - Connections with other mathematical fields
  - Practical implications

- **Conclusion**
  - Summary of findings
  - Future directions

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**Notes:**

- [1] P. v. 11
- [3] P. v. 15

---

**Figures and Diagrams:**

- [Figure 1: Rational Shi Tableaux]

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**Mathematical Equations and Notations:**

- $\alpha, \beta, \gamma, \delta$
- $\phi, \psi, \chi, \theta$
- $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$

---

**References:**

Robin Sulzgruber

On (rational) Shi tableaux

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On (rational) Shi tableaux

Robin Sulzgruber

March 2017