Difference operators for functions of partitions and its application to hook-content identities
(joint with Paul-Olivier Dehaye and Guo-Niu Han)

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Definitions

- **partition**: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$.
- **size**: $|\lambda| = \sum_{1 \leq i \leq \ell} \lambda_i$.
- **Young diagram**: boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$-th row.
- **hook length**: $h_{\Box} := \#$ boxes exactly to the right, exactly above, and $\Box$ itself.
- $H(\lambda)$: the product of all hook lengths in the Young diagram.

![Young diagram example](image)

**Figure**: The Young diagram of the partition $(6, 3, 3, 2)$ and the hook lengths of corresponding boxes.
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- **content**: \( c_\square := j - i \) for the box \( \square \) in the \( i \)-th row and \( j \)-th column.

![Diagram](attachment:image.png)

**Figure**: The contents of the partition (6, 3, 3, 2).
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- **content**: \( c_{\square} := j - i \) for the box \( \square \) in the \( i \)-th row and \( j \)-th column.
- **standard Young tableau (SYT)** of the shape \( \lambda \): fill in the Young diagram with distinct numbers 1 to \( |\lambda| \) such that the numbers in each row and each column are increasing.
- **\( f_\lambda \)**: \# SYTs of the shape \( \lambda \).

\[
\begin{array}{ccc}
6 & 9 \\
3 & 8 & 14 \\
2 & 5 & 13 \\
1 & 4 & 7 & 10 & 11 & 12
\end{array}
\]

*Figure*: A standard Young tableau of the shape \((6, 3, 3, 2)\).
RSK algorithm (Robinson-Schensted-Knuth) ⇒ \( \frac{1}{n!} \sum_{|\lambda|=n} f_{\lambda}^2 = 1. \)
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**Theorem (Nekrasov and Okounkov 2003, Westbury 2006, Han 2008)**

\[
\sum_{n \geq 0} \frac{x^n}{n!^2} \left( \sum_{|\lambda|=n} f_{\lambda}^2 \prod_{\square \in \lambda} (y + h_{\square}^2) \right) = \prod_{i \geq 1} (1 - x^i)^{-1} - y.
\]

First proved by Nekrasov and Okounkov in their study of Seiberg-Witten Theory on supersymmetric gauges in particle physics.

Another corollary is the Marked hook formula:

\[
\frac{1}{n!} \sum_{|\lambda|=n} f_{\lambda}^2 \sum_{h \in H(\lambda)} (y + h^2) = n \left( \frac{3n-1}{2} \right).
\]
Theorem (Nekrasov and Okounkov 2003, Westbury 2006, Han 2008)

\[ \sum_{n \geq 0} \frac{x^n}{n!^2} \left( \sum_{|\lambda| = n} f_{\lambda}^2 \prod_{\Box \in \lambda} (y + h_{\Box}^2) \right) = \prod_{i \geq 1} (1 - x^i)^{-1 - y}. \]

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Rediscovered independently by Westbury using D’Arcais polynomials and by Han using Macdonald’s identity.

Theorem (Han 2008)

Let \( \mathcal{H}_t(\lambda) \) be the multiset of the hook lengths of \( \lambda \) which are divisible by \( t \). Then

\[ \sum_{\lambda \in P} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left( y - \frac{tyz}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z}(1 - x^k)}. \]
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\]

- The case \( z = 0, \ y = 1 \) gives the generating function for the number of partitions.
The RSK algorithm (Robinson-Schensted-Knuth) \( \Rightarrow \frac{1}{n!} \sum_{|\lambda| = n} f_{\lambda}^2 = 1 \).

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Another corollary is the Marked hook formula:

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\frac{1}{n!} \sum_{|\lambda| = n} f_{\lambda}^2 \sum_{h \in H(\lambda)} h^2 = \frac{n(3n - 1)}{2}.
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\( \frac{f_\lambda^2}{|\lambda|!} \) is called the **Plancherel measure** of the partition \( \lambda \).

\( \frac{1}{n!} \sum_{|\lambda|=n} f_\lambda^2 g(\lambda) \) is called the **n-th Plancherel average** of the function \( g(\lambda) \).

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**Problem**

For which function \( g(\lambda) \), its Plancherel average \[ \frac{1}{n!} \sum_{|\lambda|=n} f^2_\lambda g(\lambda) \] has a nice expression?
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**Han 2008**

- \( \frac{1}{n!} \sum_{|\lambda|=n} f_{\lambda}^2 \sum_{\square \in \lambda} h_{\square}^2 = \frac{3n^2 - n}{2} \).
- \( \frac{1}{n!} \sum_{|\lambda|=n} f_{\lambda}^2 \sum_{\square \in \lambda} h_{\square}^4 = \frac{40n^3 - 75n^2 + 41n}{6} \).
- \( \frac{1}{n!} \sum_{|\lambda|=n} f_{\lambda}^2 \sum_{\square \in \lambda} h_{\square}^6 = \frac{1050n^4 - 4060n^3 + 5586n^2 - 2552n}{24} \).
Conjecture (Han 2008)

The Plancherel average of the function \( g(\lambda) = \sum_{\square \in \lambda} h_{\square}^{2k} \):

\[
P(n) = \frac{1}{n!} \sum_{|\lambda|=n} f_{\lambda}^2 \sum_{\square \in \lambda} h_{\square}^{2k}
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is always a polynomial of \( n \) for every \( k \in \mathbb{N} \).
Conjecture (Han 2008)

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is always a polynomial of $n$ for every $k \in \mathbb{N}$.

- This conjecture was proved and generalized by Stanley.

Theorem (Stanley 2010)

Let $Q_1$ and $Q_2$ be two given symmetric functions. Then the Plancherel average of the function $Q_1(h^{2}_{\square} : \square \in \lambda)Q_2(c_{\square} : \square \in \lambda)$:

$$P(n) = \frac{1}{n!} \sum_{|\lambda|=n} f^2_{\lambda} Q_1(h^{2}_{\square} : \square \in \lambda)Q_2(c_{\square} : \square \in \lambda)$$

is a polynomial of $n$.

- Olshanski (2010) also proved the content case.
An application of Han-Stanley Theorem:

**Corollary (Okada-Panova 2008)**

\[
n! \sum_{|\lambda|=n} \frac{\sum_{\square \in \lambda} \prod_{i=1}^{r} (h_{\square}^2 - i^2)}{H(\lambda)^2} = \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1} \prod_{j=0}^{r} (n-j).
\]
An application of Han-Stanley Theorem:

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\]

**Definition**

Let \( g(\lambda) \) be a function defined on partitions. The difference operator \( D \) on functions of partitions is defined by

\[
 Dg(\lambda) := \sum_{|\lambda^+ / \lambda| = 1} g(\lambda^+) - g(\lambda).
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An application of Han-Stanley Theorem:

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**Definition**

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\[ Dg(\lambda) := \sum_{|\lambda^+/\lambda|=1} g(\lambda^+) - g(\lambda). \]

The coefficient on the right hand side of Okada-Panova formula can be obtained by letting the difference operator act on one single partition:

\[ H_{\lambda} D^{r+1} \left( \frac{\sum_{\square \in \lambda} \prod_{1 \leq j \leq r} (h_{\square}^2 - j^2)}{H_{\lambda}} \right) = \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1}. \]
\[ \Delta g(x) := g(x + 1) - g(x). \]
\[ \Delta^r g(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} g(x + i). \]
\( g(x) \) is a polynomial iff \( \Delta^{r+1} g(x) = 0 \) for some \( r \).

Basis of polynomials:
\( \{ g(x) = x^k : k \in \mathbb{N} \} \).

Other posets: posets of (1) partitions, (2) partitions with the given \( t \)-core, (3) self-conjugate partitions, (4) doubled distinct partitions, (5) strict partitions?

\[ \sum_{|\lambda/\mu| = n} f_{\lambda/\mu} Q_1(h_{\square}^2 : \square \in \lambda) Q_2(c_{\square} : \square \in \lambda) \]
is a polynomial of \( n \).

\[ Dg(\lambda) := \sum_{|\lambda^+ / \mu| = 1} g(\lambda^+) - g(\lambda). \]
\[ D^n g(\mu) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \sum_{|\lambda/\mu| = k} f_{\lambda/\mu} g(\lambda). \]
\[ \sum_{|\lambda/\mu| = n} f_{\lambda/\mu} g(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu). \]
\( g(\lambda) \) is a \( D \)-polynomial iff \( D^{n+1} g(\lambda) = 0 \) for some \( n \).

Basis of \( D \)-polynomials? hard to characterize!

We show that \( \frac{Q_1(h_{\square}^2 : \square \in \lambda) Q_2(c_{\square} : \square \in \lambda)}{H_\lambda} \) is always a \( D \)-polynomial (A long and technique proof). Therefore

\[ \frac{1}{(n + |\mu|)!} \sum_{|\lambda/\mu| = n} f_{\lambda} f_{\lambda/\mu} Q_1(h_{\square}^2 : \square \in \lambda) Q_2(c_{\square} : \square \in \lambda) \]
A partition $\lambda$ is called a \textit{t-core partition} if it has no hook length $t$. 

Example:

$$D_t g((3,1)) = g((6,1)) + g((3,1,1,1,1)) + g((3,2,2)) - g((3,1))$$

$g(\lambda)$ is a $D_t$-polynomial iff $D_{r+1} t g(\lambda) = 0$ for some $r$. 

Question: which functions are $D_t$-polynomials?
The \( t \)-difference operator for function of partitions

- A partition \( \lambda \) is called a \textit{\( t \)-core partition} if it has no hook length \( t \).
- We write \( \lambda \geq_t \mu \) if \( \mu \) is obtained by removing some \( t \)-hooks from \( \lambda \).

\[(18, 7, 6) \xrightarrow{t=3} (3, 1)\]
The $t$-difference operator for function of partitions

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Let $\lambda$ be a partition and $g$ be a function defined on partitions. The $t$-difference operator $D_t$ is defined by

\[
D_t g(\lambda) := \sum_{\lambda^+ \geq_t \lambda, |\lambda^+ / \lambda| = t} g(\lambda^+) - g(\lambda).
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Main Theorem (X. 2015, joint with Dehaye and Han)

Suppose that $t$ is a positive integer, $u', v', j_u, j'_v, k_u, k'_v$ are nonnegative integers and $\mu$ is a given partition. Then for every $r > \sum_{u=1}^{u'} (k_u + 1) + \sum_{v=1}^{v'} \frac{k'_v + 2}{2}$ we have

$$D'_t \left( \frac{1}{H_t(\lambda)} \left( \prod_{u=1}^{u'} \sum_{\Box \in \lambda \atop h_\Box \equiv \pm j_u \mod t} h_\Box^{2k_u} \right) \left( \prod_{v=1}^{v'} \sum_{\Box \in \lambda \atop c_\Box \equiv j'_v \mod t} c_\Box^{k'_v} \right) \right) = 0$$

for every partition $\lambda$. Moreover,

$$P(n) := \sum_{\lambda \geq \mu \atop |\lambda/\mu| = nt} \frac{F_{\lambda/\mu}}{H_t(\lambda)} \left( \prod_{u=1}^{u'} \sum_{\Box \in \lambda \atop h_\Box \equiv \pm j_u \mod t} h_\Box^{2k_u} \right) \left( \prod_{v=1}^{v'} \sum_{\Box \in \lambda \atop c_\Box \equiv j'_v \mod t} c_\Box^{k'_v} \right)$$

is a polynomial of $n$ with degree at most $\sum_{u=1}^{u'} (k_u + 1) + \sum_{v=1}^{v'} \frac{k'_v + 2}{2}$.
The outline of the proof of the main results

**Step 1**: We construct some complicated sets $A_k(k \geq 0)$ of functions of partitions such that $g \in A_{k+1}$ implies $D_t g \in A_k$. Finally $D_t^{k+1} g = 0$ if $g \in A_k$. 
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Step 2: Let $k$ be a nonnegative integer and $0 \leq j \leq t - 1$. Then

$$\frac{1}{H_t(\lambda)} \left( \prod_{u=1}^{u'} \sum_{\square \in \lambda} h_{\square}^{2k_u} \right) \left( \prod_{v=1}^{v'} \sum_{\square \in \lambda} c_{\square}^{k'_v} \right)$$

is in the set $A_{r-1}$ for some $r$. 

$$h_{\square} \equiv \pm j_u \pmod{t}$$

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$$

is in the set $A_{r-1}$ for some $r$.

Step 3: By the above two steps we know there exists some $r \in \mathbb{N}$ such that

$$
D_t^r \left( \frac{1}{H_t(\lambda)} \left( \prod_{u=1}^{u'} \sum_{\square \in \lambda} h_{\square}^{2k_u} \right) \left( \prod_{v=1}^{v'} \sum_{\square \in \lambda} c_{\square}^{k'_v} \right) \right) = 0
$$

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Other applications for the case $t = 1$:

**Corollary**

\[
\frac{1}{(n + |\mu|)!} \sum_{|\lambda/\mu| = n} f_\lambda f_{\lambda/\mu} = \frac{1}{H(\mu)}.
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The above identity can be given a combinatorial proof by using RSK algorithm.
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**Corollary (Okada-Panova 2008)**

$$n! \sum_{|\lambda| = n} \frac{\sum_{\square \in \lambda} \prod_{i=1}^{r} (h_\square^2 - i^2)}{H(\lambda)^2} = \frac{1}{2(r + 1)^2} \binom{2r}{r} \binom{2r + 2}{r + 1} \prod_{j=0}^{r}(n - j).$$

**Corollary (Fujii-Kanno-Moriyama-Okada 2008)**

$$n! \sum_{|\lambda| = n} \frac{\sum_{\square \in \lambda} \prod_{i=0}^{r-1} (c_\square^2 - i^2)}{H(\lambda)^2} = \frac{(2r)!}{(r + 1)!^2} \prod_{j=0}^{r}(n - j).$$
Corollaries of the main theorem for general $t$.

**Corollary**

*Suppose that $\mu$ is a given $t$-core partition. Then we have*

$$
\sum_{\lambda \text{-core} = \mu} \frac{F_{\lambda/\mu}}{H_t(\lambda)} \sum_{\Box \in \lambda} h_{\Box}^2 = nt^2 + 3t \binom{n}{2}.
$$

*Furthermore,*

$$
\sum_{\lambda \text{-core} = \mu} \frac{F_{\lambda/\mu}}{H_t(\lambda)} \sum_{\Box \in \lambda} h_{\Box}^2 = \frac{3t^2n^2}{2} + \frac{nt(t^2 - 3t - 1 + 24|\mu|)}{6} + \sum_{\Box \in \mu} h_{\Box}^2.
$$

*In particular, let $\mu = \emptyset$. We have*

$$
\sum_{\lambda \text{-core} = \emptyset} \frac{n! \ t^n}{H_t(\lambda)^2} \sum_{\Box \in \lambda} h_{\Box}^2 = \frac{3t^2n^2}{2} + \frac{nt(t^2 - 3t - 1)}{6}.
$$
Motivated by Han’s proof of Nekrasov-Okounkov Formula, Pétréolle obtained the following results.

**Theorem (Pétréolle 2015)**

For any complex number $z$, the following formulas hold:

\[
\left( \prod_{i \geq 1} \frac{1 - x^{2i} z + 1}{1 - x^i} \right)^{2z - 1} = \sum_{\lambda \in SC} \delta_{\lambda} x^{||\lambda||} \prod_{h \in \mathcal{H}(\lambda)} \left( 1 - \frac{2z}{h \varepsilon_h} \right),
\]

\[
\prod_{k \geq 1} (1 - x^k)^{2z^2 + z} = \sum_{\lambda \in DD} \delta_{\lambda} x^{||\lambda||/2} \prod_{h \in \mathcal{H}(\lambda)} \left( 1 - \frac{2z + 2}{h \varepsilon_h} \right),
\]

where the sum is over all self-conjugate and doubled distinct partitions respectively.
Self-conjugate partitions

- **self-conjugate partition**: a partition whose Young diagram is symmetric along the main diagonal.
- **$\mathcal{SC}$**: the set of self-conjugate partitions.

The $t$-difference operator $D_{\mathcal{SC}}^t$ for self-conjugate partitions is defined by

$$D_{\mathcal{SC}}^t g(\lambda) := \sum_{\lambda' + \mu \in \mathcal{SC}, \lambda' + \mu \geq t} |\lambda'| \frac{|\lambda' + \mu|}{|\lambda|} = 2 \sum_{\lambda' + \mu \in \mathcal{SC}, \lambda' + \mu \geq t} g(\lambda' + \mu) - g(\lambda).$$

**Theorem (X. 2015, joint with Han)**

Let $t = 2t'$ be an even positive integer, $\mu$ be a given self-conjugate partition, and $u', v', j, v, k, k'$ be nonnegative integers. Then we have

$$P(n) = (2t')^n n! \sum_{\lambda \in \mathcal{SC}, |\lambda| = 2nt} H_t(\lambda) = 2^n Q_1(h) Q_2(c)$$

$H_t(\lambda)$ is a polynomial in $n$ for any symmetric functions $Q_1$ and $Q_2$. 

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- **SC**: the set of self-conjugate partitions.
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$$D_t^{SC} g(\lambda) := \sum_{\lambda^+ \in SC, \lambda^+ \geq t, \lambda \vdash nt} g(\lambda^+ - \lambda).$$

**Theorem (X. 2015, joint with Han)**

Let $t = 2t'$ be an even positive integer, $\mu$ be a given self-conjugate partition, and $u', v', j_u, j_v, k_u, k_v$ be nonnegative integers. Then we have

$$P(n) = (2t)^n n! \sum_{\lambda \in SC, |\lambda| = 2nt} \frac{Q_1(h^2 : h \in \mathcal{H}(\lambda)) Q_2(c : c \in \mathcal{C}(\lambda))}{H_t(\lambda)}$$

is a polynomial in $n$ for any symmetric functions $Q_1$ and $Q_2$. 
Corollary (Pétréolle 2015)

Let $t = 2t'$ be an even positive integer. Then

$$\sum_{\lambda \in SC, |\lambda| = 2nt, \# \mathcal{H}_t(\lambda) = 2n} \frac{1}{H_t(\lambda)} = \frac{1}{(2t)^n n!}.$$ 

Corollary

Let $t = 2t'$ be an even positive integer. We have

$$\sum_{\lambda \in SC, |\lambda| = 2nt, \# \mathcal{H}_t(\lambda) = 2n} H_t(\lambda) = 6t^2 n^2 + \frac{1}{3} (t^2 - 6t - 1) tn,$$

$$\sum_{h \in \mathcal{H}(\lambda)} h^2 = 6t^2 n^2 + \frac{1}{3} (t^2 - 6t - 1) tn,$$

$$\sum_{c \in \mathcal{C}(\lambda)} c^2 = 2t^2 n^2 + \frac{1}{3} (t^2 - 6t - 1) tn.$$
A strict partition (bar partition) is a finite strict decreasing sequence of positive integers $ar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_\ell)$.

The doubled distinct partition $\psi(\bar{\lambda})$ of a strict partition $\bar{\lambda}$, is the usual partition whose Young diagram is obtained by adding $\bar{\lambda}_i$ boxes to the $i$-th column of the shifted Young diagram of $\bar{\lambda}$ for $1 \leq i \leq \ell(\bar{\lambda})$.

For example, $(6, 4, 4, 1, 1)$ is the doubled distinct partition of $(5, 2, 1)$.

Figure: From strict partitions to doubled distinct partitions.
The $\mathcal{DD}$: the set of doubled distinct partitions.

The $t$-difference operator $D_t^{\mathcal{DD}}$ for doubled distinct partitions is defined by

$$D_t^{\mathcal{DD}} g(\lambda) = \sum_{\lambda^+ \in \mathcal{DD}, \lambda^+ \geq_t \lambda, |\lambda^+ / \lambda| = 2t} g(\lambda^+) - g(\lambda).$$
- $\mathcal{DD}$: the set of doubled distinct partitions.

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**Theorem (X. 2015, joint with Han)**

Let $t = 2t' + 1$ be an odd positive integer. The following summation for the positive integer $n$

$$(2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda| = 2nt \\ \# \mathcal{H}_t(\lambda) = 2n}} \frac{Q_1(h^2 : h \in \mathcal{H}(\lambda)) Q_2(c : c \in \mathcal{C}(\lambda))}{H_t(\lambda)}$$

is a polynomial in $n$ for any symmetric functions $Q_1$ and $Q_2$. 
Corollary (Pétréolle 2015)

Let \( t = 2t' + 1 \) be an odd positive integer. Then

\[
\sum_{\lambda \in \mathcal{DD}, \# \mathcal{H}_t(\lambda) = 2n, |\lambda| = 2nt} \frac{1}{H_t(\lambda)} = \frac{1}{(2t)^{n}n!}.
\]

Corollary

Let \( t = 2t' + 1 \) be an odd positive integer. We have

\[
(2t)^n n! \sum_{\lambda \in \mathcal{DD}, \# \mathcal{H}_t(\lambda) = 2n, |\lambda| = 2nt} \frac{1}{H_t(\lambda)} \sum_{h \in \mathcal{H}(\lambda)} h^2 = 6t^2n^2 + \frac{1}{3}(t^2 - 6t + 2)tn,
\]

\[
(2t)^n n! \sum_{\lambda \in \mathcal{DD}, \# \mathcal{H}_t(\lambda) = 2n, |\lambda| = 2nt} \frac{1}{H_t(\lambda)} \sum_{c \in \mathcal{C}(\lambda)} c^2 = 2t^2n^2 + \frac{1}{3}(t^2 - 6t + 2)tn.
\]
Corollary

Let $Q$ be a given symmetric function, and $\bar{\mu}$ be a given strict partition. Then

$$P(n) = \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2|\bar{\lambda}| - |\bar{\mu}| - \ell(\bar{\lambda}) + \ell(\bar{\mu}) f_{\bar{\lambda}/\bar{\mu}}}{\bar{H}(\bar{\lambda})} \bar{H}(\bar{\mu}) Q\left(\binom{\bar{\square}}{2} : \Box \in \bar{\lambda}\right)$$

is a polynomial of $n$. In particular,

$$\sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2|\bar{\lambda}| - |\bar{\mu}| - \ell(\bar{\lambda}) + \ell(\bar{\mu}) f_{\bar{\lambda}/\bar{\mu}}}{\bar{H}(\bar{\lambda})} \bar{H}(\bar{\mu}) \left(\sum_{\Box \in \bar{\lambda}} \binom{\bar{\square}}{2} - \sum_{\Box \in \bar{\mu}} \binom{\bar{\square}}{2}\right) = \binom{n}{2} + n|\bar{\mu}|.$$

Corollary

Suppose that $k$ is a given nonnegative integer. Then

$$\sum_{|\bar{\lambda}|=n} \frac{2|\bar{\lambda}| - \ell(\bar{\lambda}) f_{\bar{\lambda}}}{\bar{H}(\bar{\lambda})} \sum_{\Box \in \bar{\lambda}} \binom{\bar{\square}}{2} \frac{1}{k} = \frac{2^k}{(k+1)!} \binom{n}{k+1}.$$
Thank You for Listening!