Orbital profile and orbit algebra of oligomorphic permutation groups

Conjecture of Macpherson

Séminaire Lotharingien de Combinatoire

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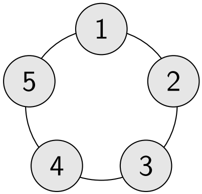
joint work with Nicolas M. Thiéry

Laboratoire de Recherche en Informatique
Université Paris-Sud

March 29th of 2017

Age and profile: example on a finite group (1)

Action of the cyclic group $G = C_5$ on the five pearl necklace
$\rightarrow$ induced action on subsets of pearls



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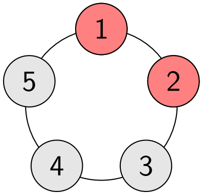

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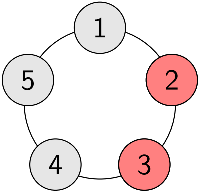

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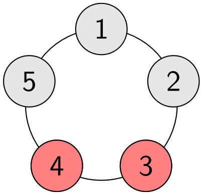

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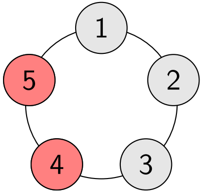

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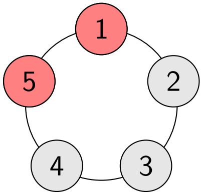

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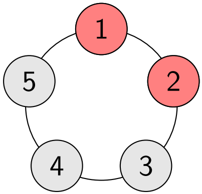

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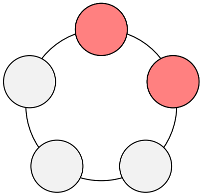

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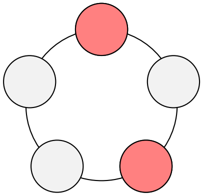

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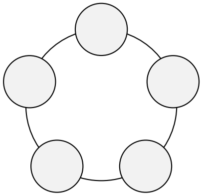

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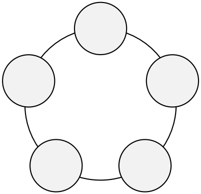

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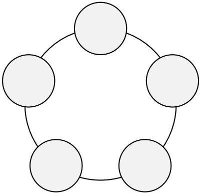

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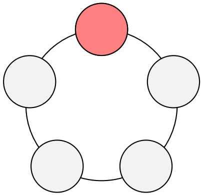

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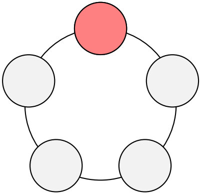

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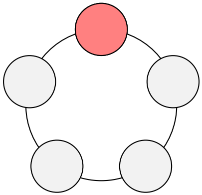

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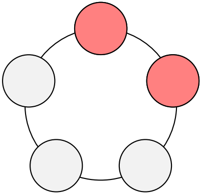

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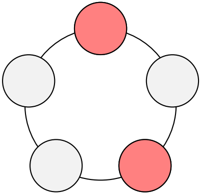

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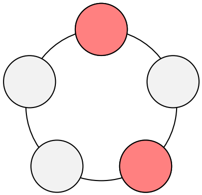

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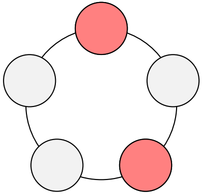

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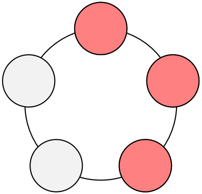

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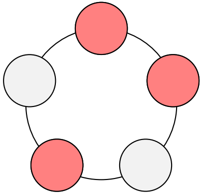

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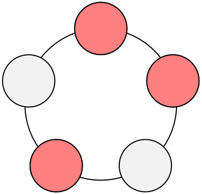

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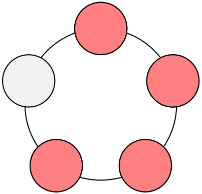

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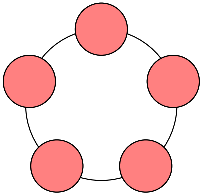

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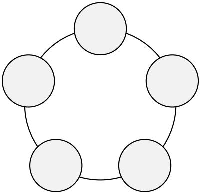

Age and profile: example on a finite group (2)

Generating polynomial of the profile:

$$\mathcal{H}_G(z) = \sum_{n \geq 0} \varphi_G(n) z^n = 1 + z + 2z^2 + 2z^3 + z^4 + z^5$$

Can be calculated straightly by Pólya's theory

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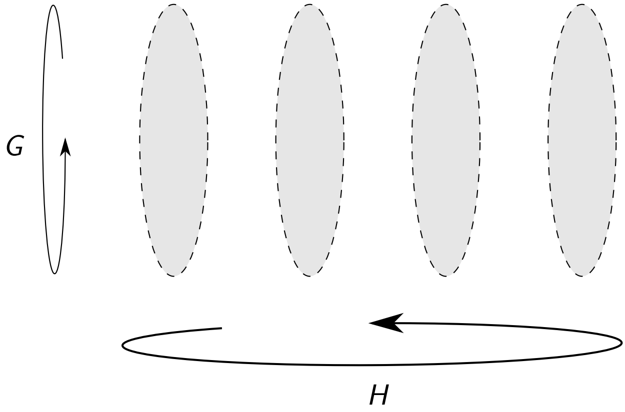
$\rightarrow$ **Oligomorphic groups**:

$$\varphi_G(n) < \infty \quad \forall n \in \mathbb{N}$$

Wreath product of two permutation groups

$G \leq \mathfrak{S}_M$, $H \leq \mathfrak{S}_N$

$G \wr H$ has a natural action on $E = \sqcup_{i=1}^{N} E_i$, with $\mathrm{card} E_i = M$.



Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)

  An orbit of degree $n \longleftrightarrow$ a partition of n

  $\varphi_G(n) = \mathscr{P}(n)$, the number of partitions of n

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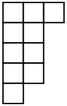
- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into parts of size $\leq m$

  $$\mathcal{H}_G = \frac{1}{\prod_{i=1}^{m}(1 - z^i)}$$

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_m$
  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into at most m parts

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  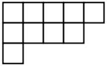

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Note

The number $\mathscr{P}(n)$ of partitions of n is neither bounded by a polynomial nor exponential.

Conjecture of Cameron

Conjecture (Cameron, 70s)

If a profile is bounded by a polynomial (thus polynomial) it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

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Note

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})} \implies \varphi_G \text{ quasi-polynomial of degree at most } k-1$$

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \ldots, x_m]$ is a graded algebra.
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Proposition

A is finitely generated $\implies$ $\text{Hilbert}(A) = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

Example

$\text{Hilbert}(\mathbb{Q}[x, y, t^3]) = \frac{1}{(1-z)^2(1-z^3)}$

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n \geq 0} A_n$ such that

$$\mathcal{H}_G = \text{Hilbert}\left(\mathbb{Q}\mathcal{A}(G)\right)$$

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- Try to show that $\mathbb{Q}\mathcal{A}(G)$ is finitely generated
- Deduce:

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$$

and thus the quasi-polynomiality of $\varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n \geq 0} A_n$
- $\dim(A_n) = \varphi_G(n)$

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- finite formal linear combinations of orbits (ex: $2o_1 + 5o_2 - o_3$)
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Product?

- Defined on subsets:

$$ef = \begin{cases} e \cup f & \text{if } e \cap f = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- $o = \{e_1, e_2, \dots\} \quad \longleftrightarrow \quad e_1 + e_2 + \cdots$

Example of product on a finite case

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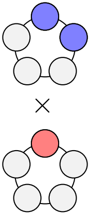

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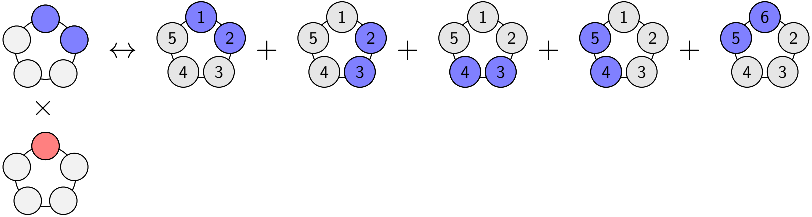

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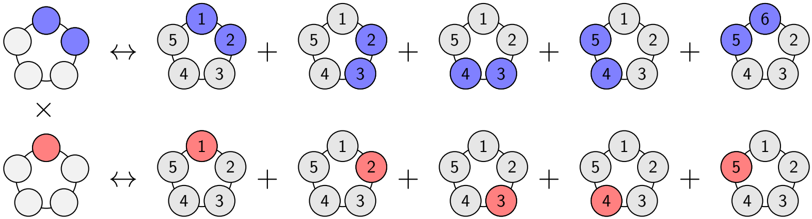

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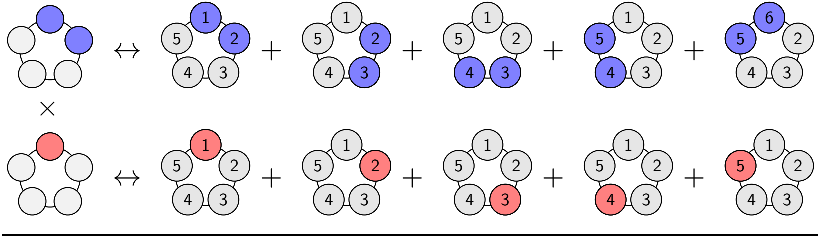

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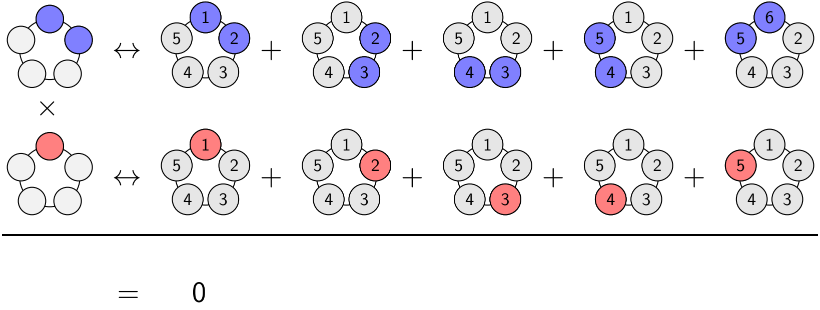

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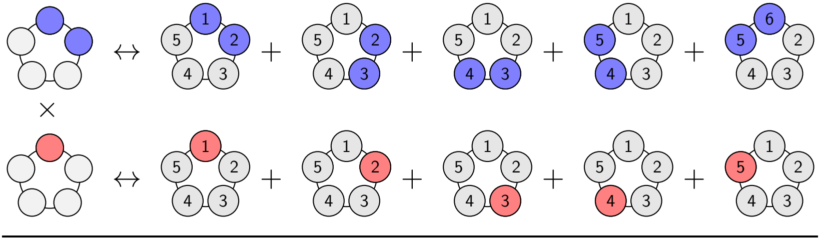

$$= \quad 0 \quad + \quad 0$$

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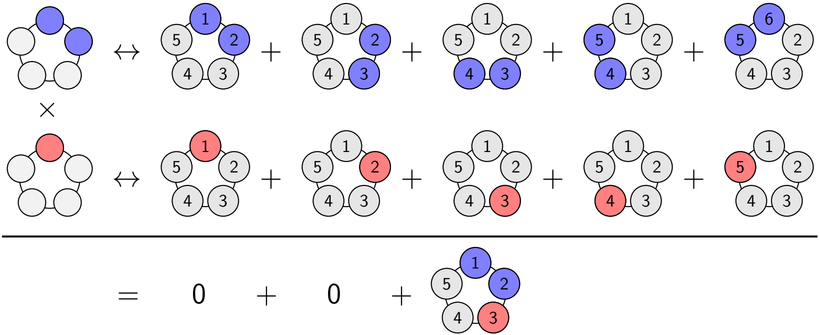

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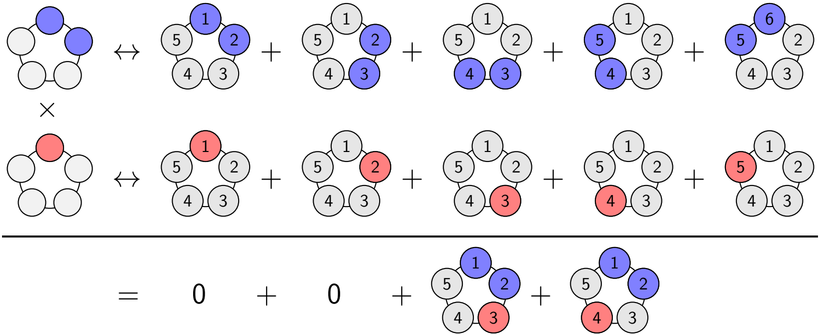

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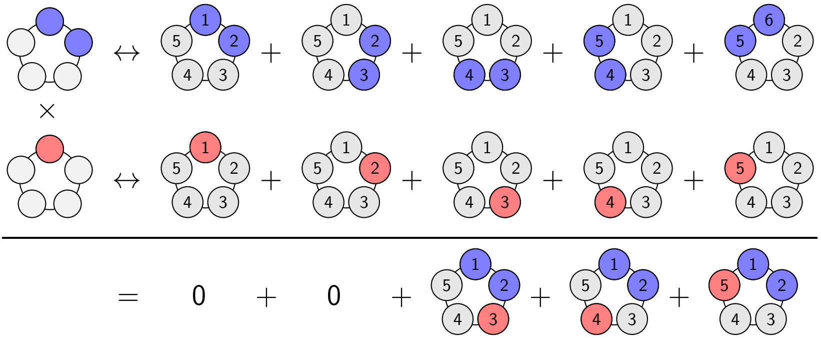

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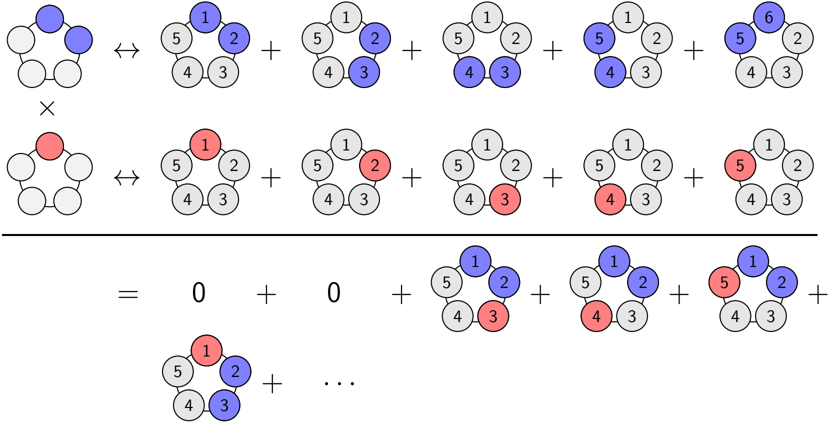

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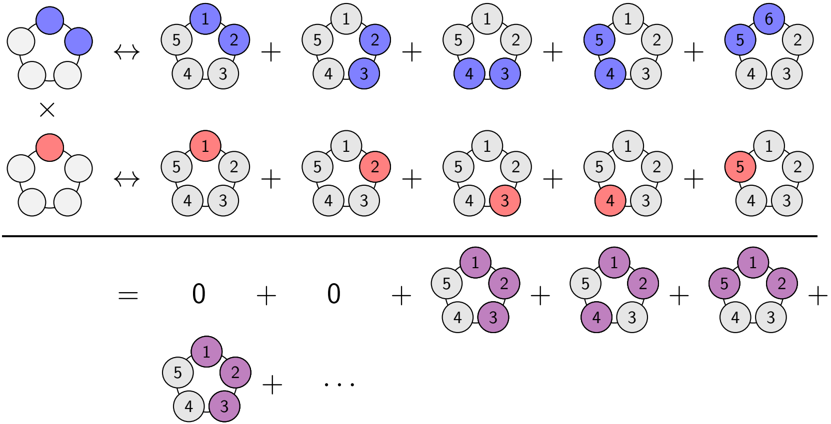

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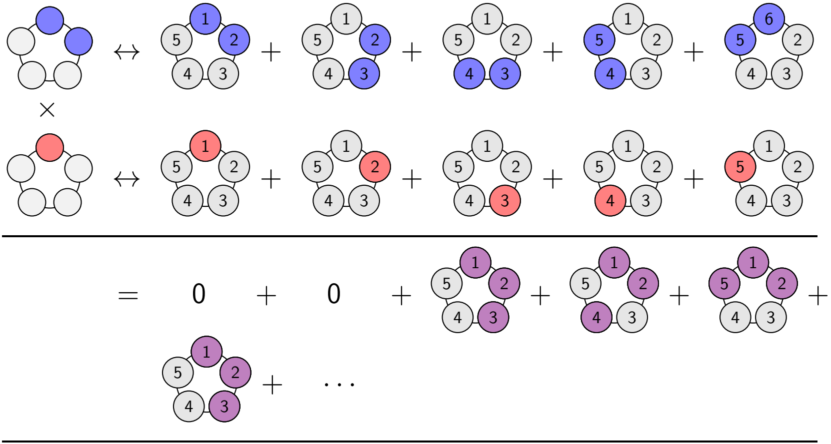

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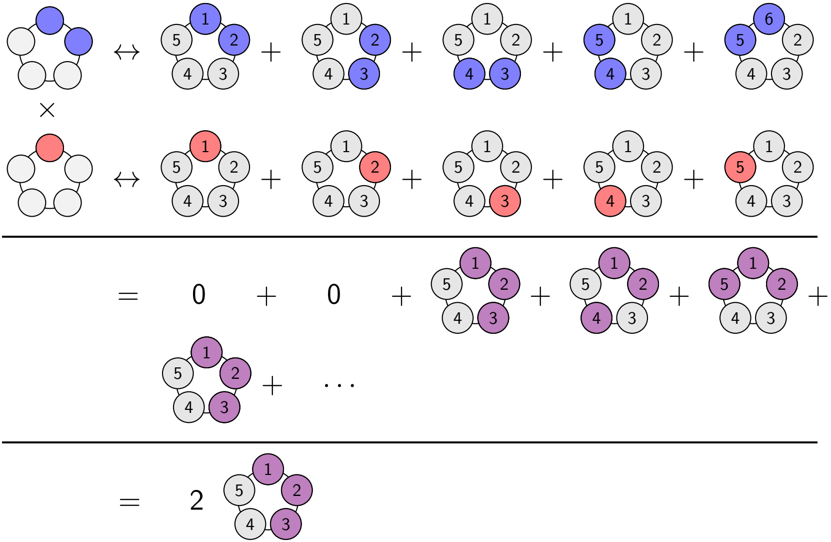

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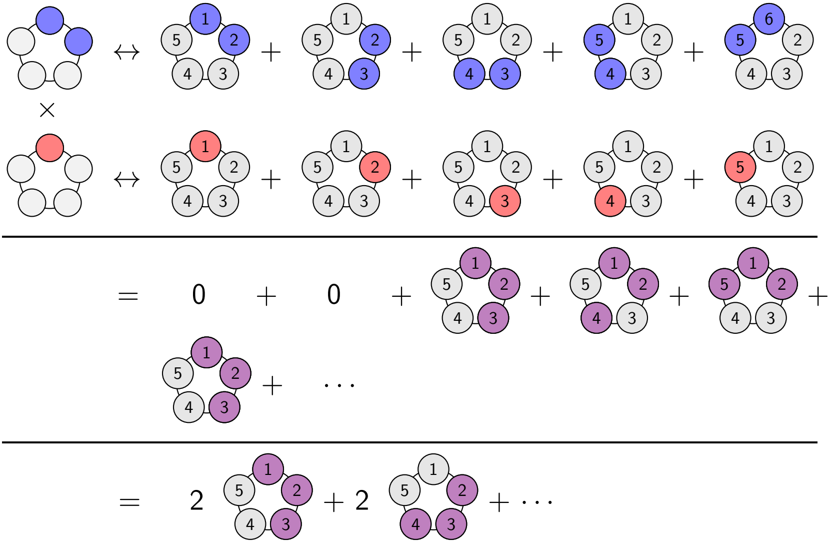

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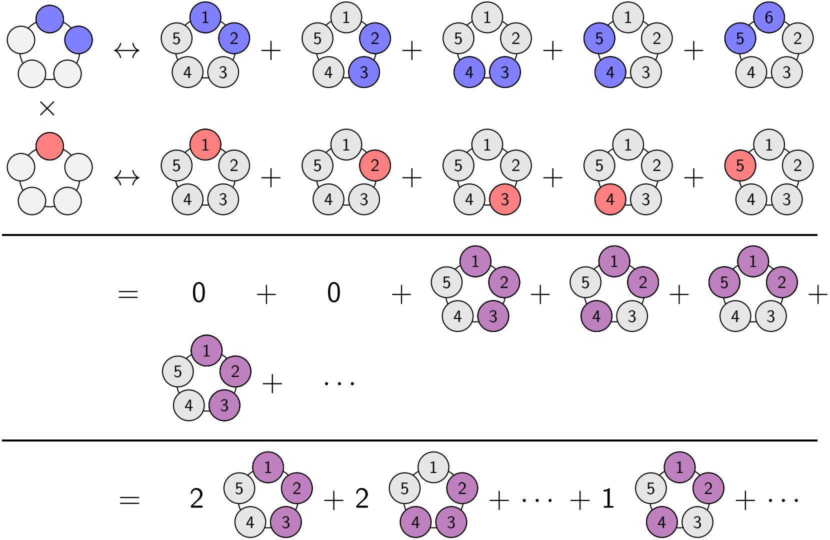

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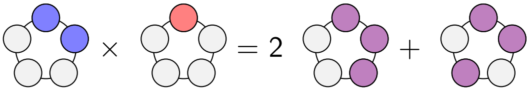
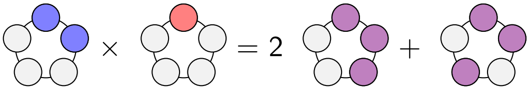


In the end: 

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Product well defined (and graded) on the space of orbits.

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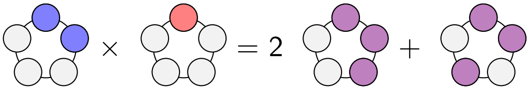

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$\longrightarrow$ **The orbit algebra of a permutation group**

Examples of orbit algebras (1)

Example 1

If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

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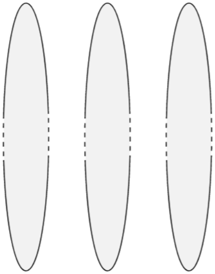

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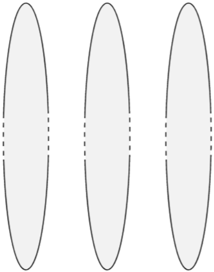

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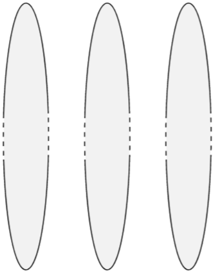

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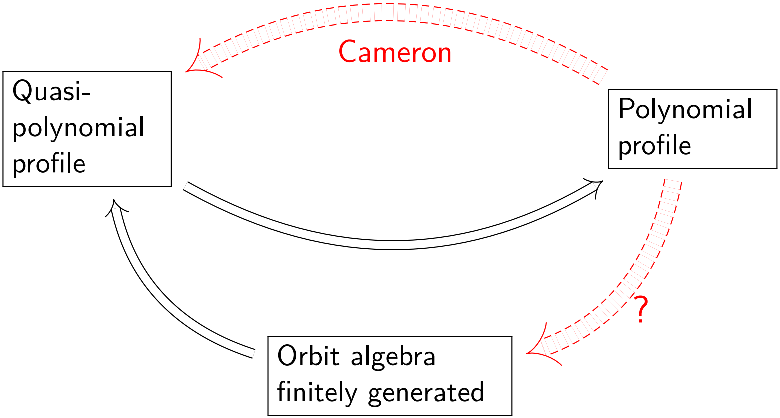
More generally, for H subgroup of $\mathfrak{S}_m$, $\mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr H) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

Overview and conjecture of Macpherson



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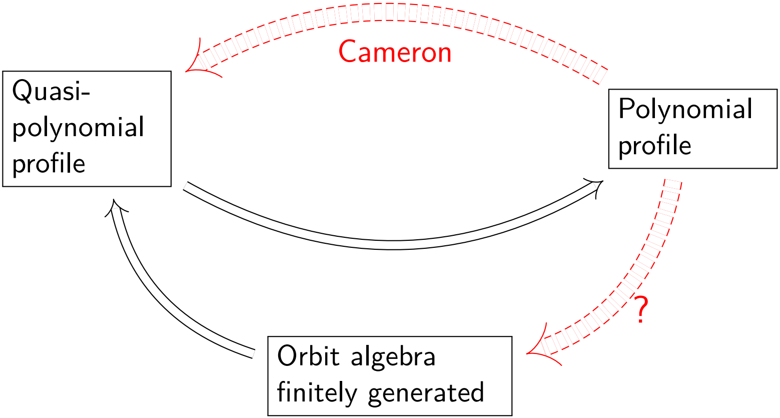

Conjecture (Macpherson, 1985)

Profile of G polynomial $\Longleftrightarrow$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

Tools

- Block structure: a partition of E such that each part is globally mapped to another one, or itself (see previous examples)
- Knowledge of algebras of wreath products
- Embedding

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- Classification of groups of profile 1 (Cameron)
- Goursat's lemma (subdirect product)

  $\Longrightarrow$ information on the age

Macpherson for bounded profiles

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- First proof by Pouzet
- By reduction, one can assume G is one of the five primitive groups (with polynomial profile)

  $\rightarrow$ orbit algebra $= \mathbb{K}[x]$
- Without reduction (constructive proof):
  $\rightarrow$ same age as $\mathfrak{S}_\infty \times G'$, G' a finite group determined by G
  $\rightarrow$ generating series: $\frac{P(z)}{(1-z)}$,
  where $P(z)$ is the generating polynomial of G'

Macpherson for linear profiles

Two essential cases

- 2 infinite orbits without blocks
- an infinity of blocks of size 2

Macpherson for linear profiles

Two essential cases

- 2 infinite orbits without blocks
- an infinity of blocks of size 2

$\rightarrow$ The conjectures of Macpherson and Cameron hold.

Context

- G: permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- **Hypothesis**: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): quasi-polynomiality of φ_G
- Conjecture (Macpherson): finite generation of the orbit algebra

Results

- Block structure of $G \Longrightarrow$ minoration of φ_G
- Lemmas and reductions $\Longrightarrow$ bounded and linear cases

Conjectures / intuition

- The orbit algebra is of Cohen-Macaulay
- The growth of the profile is determined by the block structure
- Very rigid: very few groups; classification?

Last-minute message from a very kind person

Jean-Yves,

Pour une source toujours renouvelée d'inspiration,

Pour une étoile qui brille et propose un cap,

mais éclaire tout autant de sa bienveillance

les marins d'eaux douces sur leurs eaux de traverses,

Pour cet endroit si spécial qu'est Marne-la-Vallée,

Un grand merci du fond du coeur!

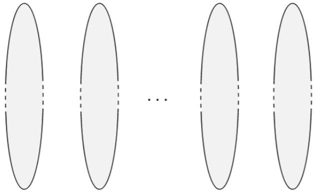
Nicolas

Examples of orbit algebras (2)

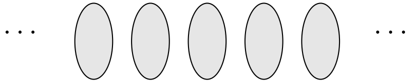
More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \dots, x_m]^H$, the algebra of invariants of H

  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.



- $G = H \wr \mathfrak{S}_\infty$:
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



Block systems

Definition: Block system

Partition of E such that each part is globally mapped to another one (or itself) by every element of G

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If G is **primitive** (i.e. admits no non trivial block system) then G has its profile equal to 1 or exponential.

$\rightarrow$ The groups we are interested in have a constanly equal to 1 profile or have a block system.

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

The complete primitive groups

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There are exactly 5 complete groups of constantly equal to 1 profile.

- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain (increasing functions)
- $\mathrm{Rev}(\mathbb{Q})$: generated by $\mathrm{Aut}(\mathbb{Q})$ and one reflection
- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
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- $\mathfrak{S}_\infty$: the symmetric group

Lower bound on the profile

Proposition

If G has either a system of M infinite blocks or an infinity of blocks of size M, then $\varphi_G(n)$ grows at least as fast as n^{M-1}.

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$\rightarrow$ Use this and the fact that the growth of the profile is at least the sum of the growths on each orbit taken separately

Finite index subgroups

Theorem

Let H be a finite index subgroup of G.

- The profiles of G and H are equivalent
- $\mathbb{Q}\mathcal{A}(H)$ finitely generated $\implies$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

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Application: reduction of Macpherson's conjecture

Without loss of generality, we may assume that G has no

- finite orbit
- orbit that split into infinite blocks

Synchronization

Case of 2 infinite orbits

$E_1 \sqcup E_2$, $\quad G_{|E_1} = G_1, G_{|E_2} = G_2$

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$\rightarrow$ very few possibilities

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Example

If $G_1 = G_2 = \mathfrak{S}_\infty$, the actions are either independant or totally synchronized.