A sextuple equidistribution arising in Pattern Avoidance

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Eulerian polynomials

Definition

The Eulerian polynomial $A_n(t)$ may be defined by Euler's basic formula (Leonhard Euler 1755):

$$\sum_{k \geq 0} (k + 1)^n t^k = \frac{A_n(t)}{(1 - t)^{n+1}}.$$

$A_1(t) = 1$
$A_2(t) = 1 + t$
$A_3(t) = 1 + 4t + t^2$
$A_4(t) = 1 + 11t + 11t^2 + t^3$
$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$
\( \mathcal{S}_n \): Set of permutations of \([n] := \{1, 2, \cdots, n\}\)

**Definition**

For \( \pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n \):

\[
\text{DES}(\pi) := \{i \in [n - 1] : \pi_i > \pi_{i+1}\}
\]

\[
\text{des}(\pi) := |\text{DES}(\pi)| \quad (\text{Descent number}).
\]

\[
\text{DES}(3.15.24) = \{1, 3\}
\]
\( \mathcal{G}_n \): Set of permutations of \([n] := \{1, 2, \cdots, n\} \)

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\]

\[
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\]

**Theorem (Riordan 1958)**

\[
A_n(t) = \sum_{\pi \in \mathcal{G}_n} t^{\text{des}(\pi)}.
\]
Inversion sequences: \( \mathcal{I}_n = \{(e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n : 0 \leq e_i < i\} \)

\( \mathcal{I}_3 = \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 1, 1), (0, 1, 2)\} \)

**Definition**

For \( e = (e_1, e_2, \ldots, e_n) \in \mathcal{I}_n \):

\[
\text{ASC}(e) := \{i \in [n - 1] : e_i < e_{i+1}\}
\]

\[
\text{asc}(e) := |\text{ASC}(e)| \quad (\text{Ascent number}).
\]

\[
\text{ASC}(0, 1, 1, 2, 0) = \{1, 3\}
\]
A natural bijection: \( \text{inv-code} \)

\[
|\mathcal{S}_n| = |\mathcal{I}_n| = n!
\]

and more...

\[
\sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)} = \sum_{e \in \mathcal{I}_n} t^{\text{asc}(e)}
\]
A natural bijection: \( \text{inv-code} \)

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\sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)} = \sum_{e \in \mathcal{I}_n} t^{\text{asc}(e)}
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A natural bijection (\textit{inv-code}) \( \phi: \mathcal{S}_n \to \mathcal{I}_n \) with \( \phi(\pi) = (e_1, \ldots, e_n) \), where

\[
e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.
\]
A natural bijection: \textit{inv-code}

\[
|\mathcal{S}_n| = |\mathcal{I}_n| = n!
\]

and more...

\[
\sum_{\pi \in \mathcal{S}_n} t^{\text{des}}(\pi) = \sum_{e \in \mathcal{I}_n} t^{\text{asc}}(e)
\]

A natural bijection (\textit{inv-code}) \( \phi : \mathcal{S}_n \to \mathcal{I}_n \) with \( \phi(\pi) = (e_1, \ldots, e_n) \), where

\[
e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.
\]

This proves even more:

\[
\sum_{\pi \in \mathcal{S}_n} t^{\text{DES}}(\pi) = \sum_{e \in \mathcal{I}_n} t^{\text{ASC}}(e),
\]

where \( t\{i_1, \ldots, i_k\} := t_{i_1} \cdots t_{i_k} \).
**Double Eulerian statistics**

**dist(e):** number of distinct positive entries in e

**Theorem (Dumont 1974)**

\[
\sum_{\pi \in S_n} t^{\text{des}(\pi)} = \sum_{e \in I_n} t^{\text{dist}(e)}.
\]
Double Eulerian statistics

\( \text{dist}(e) \): number of distinct positive entries in \( e \)

**Theorem (Dumont 1974)**

\[
\sum_{\pi \in S_n} t^{\text{des}(\pi)} = \sum_{e \in \mathcal{I}_n} t^{\text{dist}(e)}.
\]

Via **V-code** and **S-code**:

**Theorem (Foata 1977)**

\[
\sum_{\pi \in S_n} s^{\text{des}(\pi^{-1})} t^{\text{DES}(\pi)} = \sum_{e \in \mathcal{I}_n} s^{\text{dist}(e)} t^{\text{ASC}(e)}.
\]

- Rediscovered by **Visontai** (2013)
- An essentially different proof by **Aas** in **PP 2013** (Paris)
Double Eulerian polynomials (Carlitz-Roselle-Scoville 1966):

\[ A_n(s, t) := \sum_{\pi \in \mathcal{S}_n} s^{\text{des}(\pi^{-1})} t^{\text{des}(\pi)}. \]

Conjectured by Gessel (2005):

**Theorem (L. 2015)**

The integers $\gamma_{n,i,j}$ are nonnegative in:

\[ A_n(s, t) = \sum_{i,j \geq 0, \ j+2i \leq n-1} \gamma_{n,i,j}(st)^i(1+st)^j(s+t)^{n-1-j-2i}. \]
Permutations without double descents

Figure: Foata-Strehl actions on 34862571

NDD\(_n\): set of all permutations in \(S_n\) without double descents

Theorem (Foata & Schützenberger 1970)

\[
A_n(t) = \sum_{i=0}^{\lfloor(n-1)/2\rfloor} \gamma_{n,i} t^i (1 + t)^{n+1-2i},
\]

where \(\gamma_{n,i} = \#\{\pi \in \text{NDD}_n : \text{des}(\pi) = i\}\).
Permutations without double descents

$\text{NDD}_n$ : set of all permutations in $\mathfrak{S}_n$ without double descents

**Theorem (Foata & Schützenberger 1970)**

$$A_n(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i} t^i (1 + t)^{n+1-2i},$$

where $\gamma_{n,i} = \#\{\pi \in \text{NDD}_n : \text{des}(\pi) = i\}$.

**Problem**

*Is there any combinatorial interpretation for $\gamma_{n,i,j}$?*
Separable permutations

Restrict to the terms without \( s + t \):

\[ \pi = 2413 \]

\[ \text{des}(\pi) = 1 \]
\[ \text{des}(\pi^{-1}) = 2 \]

First \( \text{des}(\pi) \neq \text{des}(\pi^{-1}) \)

Definition

Permutations that avoid both the patterns 2413 and 3142 are separable permutations.

West (1995): \( |S_n(2413, 3142)| = S_n \), the \( n \)th Large Schröder numbers.
Separable permutations

\[ \leftrightarrow \]

“di-sk” trees

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Descent polynomial on Separable permutations

Via combinatorial approach using “di-sk” trees:

Theorem (Fu-L.-Zeng 2015)

\[ \sum_{\pi \in S_n(2413,3142)} t^{\text{des} (\pi)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} t^k (1 + t)^{n-1-2k}, \]

where

\[ \gamma_{n,k}^S = |\{ \pi \in S_n(3142,2413) \cap \text{NDD}_n : \text{des} (\pi) = k \}|. \]
021-avoiding inversion sequences

021-avoiding ⇔ positive entries are weakly increasing

Via bijections with “di-sk” trees:

Theorem (Fu-L.-Zeng & Corteel et al. 2015)

\[ \sum_{\pi \in S_n(2413,3142)} t^{\text{des}(\pi)} = \sum_{e \in J_n(021)} t^{\text{asc}(e)}. \]

Problem

\[ \sum_{e \in J_n(021)} t^{\text{asc}(e)} = \left\lfloor \frac{(n-1)}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{(n-1)}{2} \right\rfloor} \gamma^S_{n,k} t^k (1 + t)^{n-1-2k} \]

What is the combinatorial interpretation of \( \gamma^S_{n,k} \) in terms of 021-avoiding inversion sequences?
Double Eulerian equidistribution

Theorem (Foata 1977)
\[ \sum_{\pi \in S_n} s^{\text{des}(\pi^{-1})} t^{\text{DES}(\pi)} = \sum_{e \in I_n} s^{\text{dist}(e)} t^{\text{ASC}(e)}. \]

Restricted version of Foata’s 1977 result:

Theorem (Kim-L. 2016)
\[ \sum_{\pi \in S_n(2413, 4213)} s^{\text{des}(\pi^{-1})} t^{\text{DES}(\pi)} = \sum_{e \in I_n(021)} s^{\text{dist}(e)} t^{\text{ASC}(e)}. \]

Neither Foata’s original bijection nor Aas’ approach could be applied to prove this restricted version.
As $S_n(2413, 4213)$ is invariant under Foata-Strehl action:

**Corollary**

$$
\sum_{e \in \mathcal{I}_n(021)} t^{\text{asc}(e)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}^S t^k (1 + t)^{n-1-2k},
$$

where

$$
\gamma_{n,k}^S = |\{ e \in \mathcal{I}_n(021) : e \ has \ no \ double \ ascents, \ asc(e) = k \}|.
$$
Second application

Figure: The outline of an inversion sequence

\[ S = S(s, t; z) := \sum_{n \geq 1} z^n \sum_{\pi \in S_n(2413, 4213)} s^{\text{des}(\pi^{-1})} t^{\text{des}(\pi)} \]

**Theorem (Double Eulerian distribution)**

\[ S = t(z(s - 1) + 1)S + tz(2s - 1)S^2 + z(ts + 1)S + z. \]
Ascents on Schröder paths

A Schröder \( n \)-path is a lattice path on the plane from \((0, 0)\) to \((2n, 0)\), never going below \(x\)-axis, using the steps

\[(1, 1) \ (1, -1) \ (2, 0).\]

**Corollary (Conjecture of Corteel et al. 2015)**

An ascent in a Schröder path is a maximal string of consecutive up steps. Denoted by \(SP_n\) the set of Schröder \(n\)-path and by \(\text{asc}(p)\) the number of ascents of \(p\). Then,

\[
\sum_{e \in \mathbb{I}_n(021)} s^{\text{dist}(e)} = \sum_{p \in SP_{n-1}} s^{\text{asc}(p)}.
\]
A sextuple equidistribution (Statistics)

For each $\pi \in \mathcal{S}_n$:

- $\text{VID}(\pi) := \{2 \leq i \leq n : \pi_i$ appears to the right of $(\pi_i + 1)\}$, the values of inverse descents of $\pi$;
- $\text{LMA}(\pi) := \{i \in [n] : \pi_i > \pi_j$ for all $1 \leq j < i\}$, the positions of left-to-right maxima of $\pi$;
- $\text{LMI}(\pi) := \{i \in [n] : \pi_i < \pi_j$ for all $1 \leq j < i\}$, the positions of left-to-right minima of $\pi$;
- $\text{RMA}(\pi) := \{i \in [n] : \pi_i > \pi_j$ for all $j \geq i\}$, the positions of right-to-left maxima of $\pi$;
- $\text{RMI}(\pi) := \{i \in [n] : \pi_i < \pi_j$ for all $j \geq i\}$, the positions of right-to-left minima of $\pi$;
and for each $e \in \mathcal{I}_n$:

- $\text{DIST}(e) := \{2 \leq i \leq n : e_i \neq 0 \text{ and } e_i \neq e_j \text{ for all } j > i\}$, the positions of the last occurrence of distinct positive entries of $e$;
- $\text{ZERO}(e) := \{i \in [n] : e_i = 0\}$, the positions of zeros in $e$;
- $\text{EMA}(e) := \{i \in [n] : e_i = i - 1\}$, the positions of the entries of $e$ that achieve the maximum;
- $\text{RMI}(e) := \{i \in [n] : e_i < e_j \text{ for all } j \geq i\}$, the positions of right-to-left minima of $e$. 
A sextuple equidistribution (Main result)

Theorem (Kim-L. 2016)

There exists a bijection $\Psi : \mathcal{I}_n(021) \to \mathfrak{S}_n(2413, 4213)$, which transforms the sextuple

$$(\text{DIST, ASC, ZERO, EMA, RMI, EXPO})$$

to

$$(\text{VID, DES, LMA, LMI, RMA, RMI}).$$
The algorithm $\Psi$

The labeling algorithm, where temporary variables $L, H, P$ correspond to words *label, height, position*, works as follows:

1. (Start) $L \leftarrow 1$ (This means that 1 is assigned to $L$); draw the diagonal (line) $y = x$ on $d(e)$ and label the highest east step touched by the diagonal, say $E_k$, with $L$; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (2), if $E_P$ is a red east step (i.e. $k = 1$), otherwise go to (3);

2. draw the leftmost new line that touches at least one unlabeled black east step or a *labelable* red east step; label the highest east step touched by this new line, say $E_k$, with $L$; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (2), if $E_P$ is a red east step, otherwise go to (3);

3. go to (5), if there is a black east step $E_j$ with $j > P$ and height $d_j = H$, otherwise go to (4);

4. move from $E_P$ along the two-colored Dyck path $d(e)$ to the left and along the lines that were already drawn to the southwest until we arrive at the first unlabeled east step that is a black step or a *labelable* red step, say $E_k$; label $E_k$ with $L$; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (2), if $E_P$ is a red east step, otherwise go to (3);

5. draw the leftmost line beginning at an east step right to $E_P$ which touches at least one black east step; label the highest east step touched by this new line, say $E_k$, with $L$; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (3).
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The algorithm $\Psi$ (An example)

Figure: An example of the algorithm $\Psi$
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Merci pour votre attention