The serpent nest conjecture on accordion complexes

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78ème Séminaire lotharingien de combinatoire  
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Accordion dissections

**Dissection** = set of pairwise noncrossing diagonals

**Triangulation** = inclusion maximal dissection
Accordion dissections

Cell = bounded conn. comp. of the complement

Triangulation = all cells are triangles
Accordion dissections

Consider interlaced red and blue polygons
Accordion dissections

Fix a reference red dissection $D_0$
Accordion dissections

\[ D_\circ \text{-accordion diagonal} = \text{blue diagonal crossing a connected set of red diagonals} \]
Accordion dissections

$D_\circ$-accordion diagonal $= \text{blue diagonal crossing a connected set of red diagonals}$

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Accordion dissections

Maximal $D_6$-accordion dissection = inclusion max. dissection containing blue diagonals
**Accordion dissections**

Maximal $D_6$-accordion dissection  
“blue dissection”  
= inclusion max. dissection containing blue diagonals
Maximal $D_9$-accordion dissection

"blue dissection"  = inclusion max. dissection containing blue diagonals
Accordion dissections

\(D_\circ\) is a triangulation
Accordion dissections

$D_0$ is a triangulation $\implies$ blue dissection $=$ blue triangulations

$C_n := \frac{1}{n+1} \binom{2n}{n}$
History

Baryshnikov, *On Stokes sets* (2001)
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Chapoton, *Stokes posets and serpent nests* (2016)

Are Stokes posets lattices?

Are Stokes complexes realizable as polytopes?

\#(elements of Stokes posets) = \#(serpent nests)?
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Serpent nests

**Dual tree** $D_0^*$ of $D_0 = \text{vertices: cells of } D_0$

edges: internal diagonals of $D_0$
Serpent nests

**Serpent of** $D_\circ = \text{nonempty undirected dual path in } D^*_\circ$

crossing a connected set of diagonals
**Serpent nests**

**Serpent of** $D_\circ = \text{nonempty undirected dual path in } D^*_\circ$ crossing a connected set of diagonals

![Diagram of Serpent nests](image)
Serpent nests

**Serpent nest of** $D_0 = \text{set of serpents of } D_0 \text{ with some conditions:**}
Serpent nests

Serpent nest of $D_\circ = \text{set of serpents of } D_\circ \text{ with some conditions:}$
Serpent nests

Serpent nest of $D_\circ = \text{set of serpents of } D_\circ \text{ with three conditions: no crossing}$
Serpent nests

Serpent nest of $D_\circ = \text{set of serpents of } D_\circ \text{ with three conditions: no crossing, no common arrival}$
Serpent nests

Serpent nest of $D_6 = \text{set of serpents of } D_6 \text{ with three conditions: no crossing, no common arrival}$
**Serpent nests**

**Serpent nest of** $D_0 = \text{set of serpents of } D_0 \text{ with three conditions: no crossing, no common arrival}
Serpent nests

Serpent nest of $D_\circ = \text{set of serpents of } D_\circ \text{ with three conditions: no crossing, no common arrival, no over heading}$
D₀ is a comb triangulation $\iff$ serpent nests $= \text{noncrossing partitions} \left( C_n \right)$
Serpent nests

$D_o$ is a comb triangulation $\implies$ serpent nests $=$ noncrossing partitions ($C_n$)
Theorem (M. 2017+) 

For any dissection $D_\circ$, 

$\#(\text{maximal } D_\circ\text{-accordion dissections}) = \#(\text{serpent nests of } D_\circ)$
Catalan-like bijection

\[ C_{n+1} = \sum_{k=0}^{n} C_k \times 1 \times C_{n-k} \]
Proof: induction on \(\#(\text{diagonals of } D_\circ)\)
Proof: induction on $\#(\text{diagonals of } D_o)$
Proof: induction on $\#(\text{diagonals of } D_o)$
Proof: \{ maximal $D_6$-accordion dissections \} $\rightarrow$ \{ serpent nests of $D_6$ \}
Proof: there exists $x_\bullet \in [6_\bullet, 28_\bullet]$ such that $\{(2_\bullet, x_\bullet), (4_\bullet, x_\bullet)\} \subseteq D_\bullet$. 
Proof: separate $D$ according to $x$. 

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Proof: separate $D$ according to $x$. 

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Proof: apply the bijections obtained inductively on each side
Proof: unfold the serpents
Proof: unfold the serpents
Proof: consider red diagonals crossed by both $(2, x)$ and $(4, x)$.
Proof: keep only disconnecting diagonals (zigzag)
Proof: insert the serpent from \((1°, 3°)\) to the furthest possible one.
Proof: insert the serpent from \((1_\circ, 3_\circ)\) to the furthest possible one

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Proof: insert the serpent from \((1\circ, 3\circ)\) to the furthest possible one
Proof: insert the serpent from $(1°, 3°)$ to the furthest possible one
Proof: \{\text{serpent nests of } D_\circ\} \rightarrow \{\text{maximal } D_\circ\text{-accordion dissections}\}
Proof: validly extend $S$ around successive pivots
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Proof: validly extend $S$ around successive pivots
Proof: validly extend $S$ around successive pivots
Proof: separate according to $\pi$. 

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Proof: apply reverse bijections inductively obtained
Proof: apply reverse bijections inductively obtained
Proof: glue back together
Proof: glue back together

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Theorem (M. 2017$^+$)

For any dissection $D_o$, 
$\#(\text{maximal } D_o\text{-accordion dissections}) = \#(\text{serpent nests of } D_o)$
THANK YOU FOR YOUR KIND LISTENING!

Quessssssssstions?