Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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(Domaine St. Jacques, Ottrott)

Outline

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Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}([k])$ has at least k elements
- $\mathbb{PF}_n$ .. set of all parking functions of length n

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- $\mathbb{PF}_3$:

$$
\begin{array}{cccccc}
(1,1,1) & (1,1,2) & (1,1,3) & (1,2,1) & (1,2,2) & (1,2,3)\\
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Theorem (Folklore)

For $n \geq 0$, the cardinality of $\mathbb{PF}_n$ is $(n+1)^{n-1}$.

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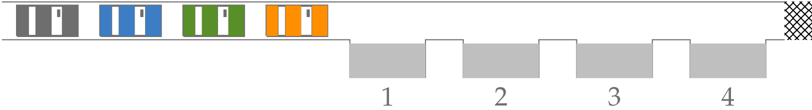

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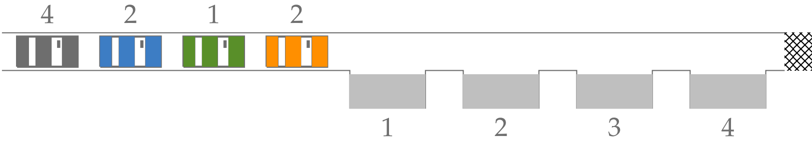

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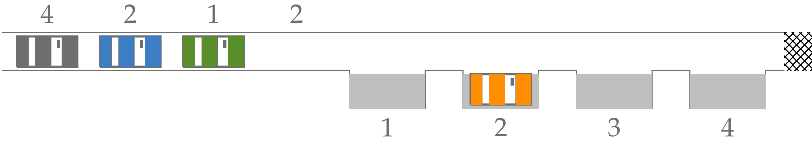

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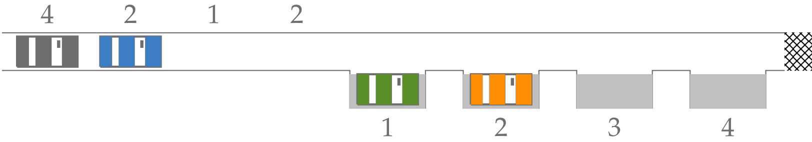

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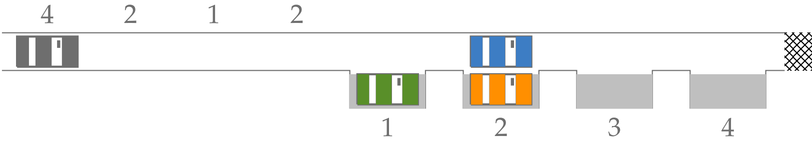

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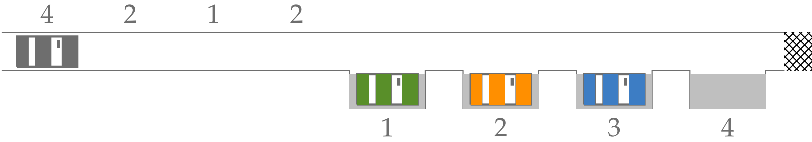

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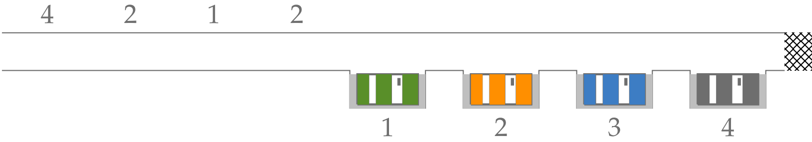

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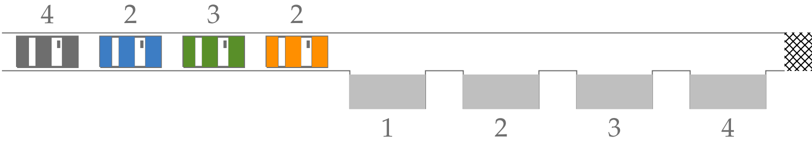

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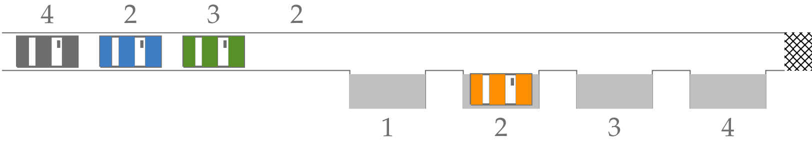

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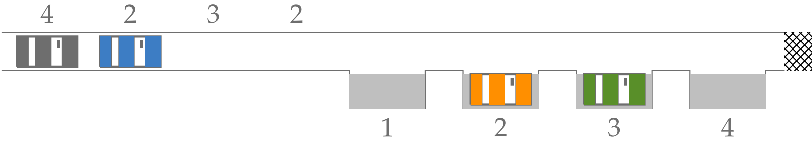

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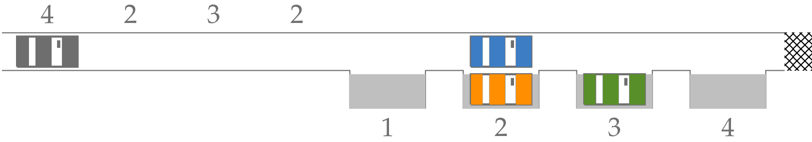

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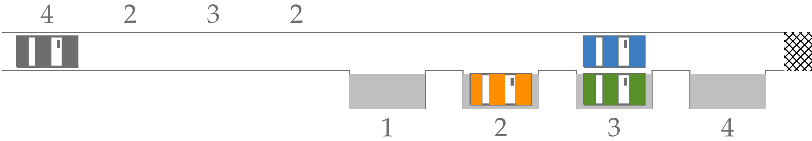

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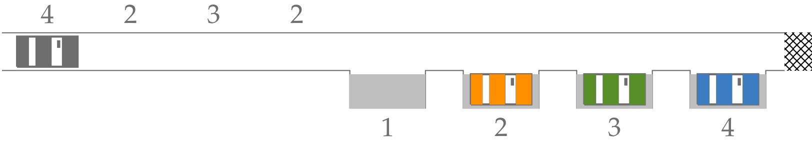

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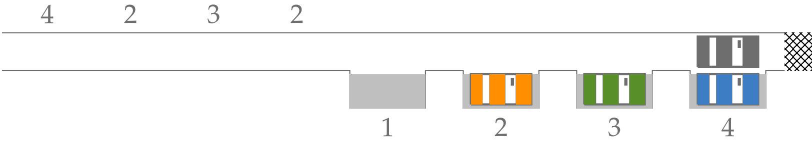

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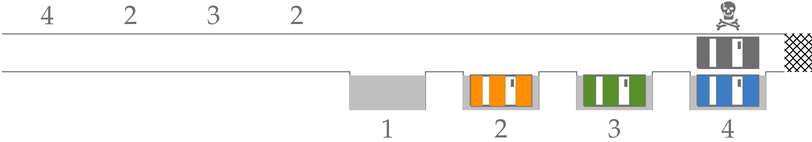
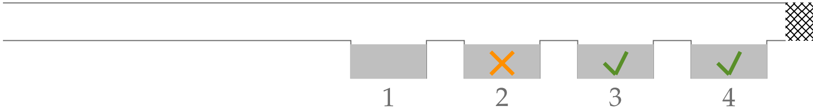


Undesired Parking Spaces



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- ***k*-avoiding parking function**: $f \in \mathbb{PF}_n$ with $k \notin f$, but $l \in f$ for all $l > k$
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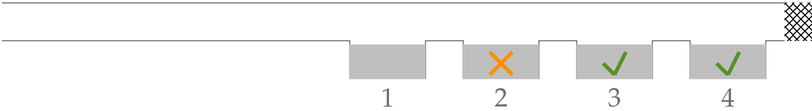

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Proposition (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $n \geq 0$ and $k \in [n]$, the cardinality of $\mathbb{PF}_{n,k}$ is

$$\frac{n!}{k!}\left|\mathbb{PF}_{k,k}\right|.$$

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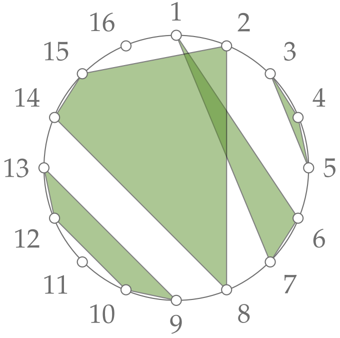

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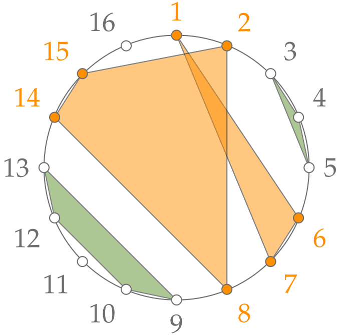

Nope!

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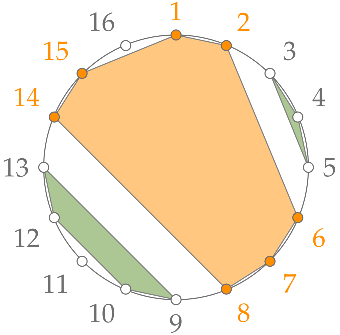

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Noncrossing Partitions

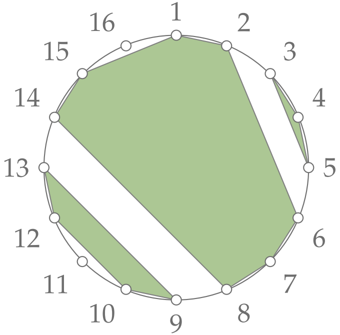
Theorem (G. Kreweras, 1972)

For $n \geq 0$, the cardinality of NC_n is

$$Cat(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Noncrossing Partitions

- **dual refinement order** $\rightsquigarrow \leq_{\mathrm{dref}}$



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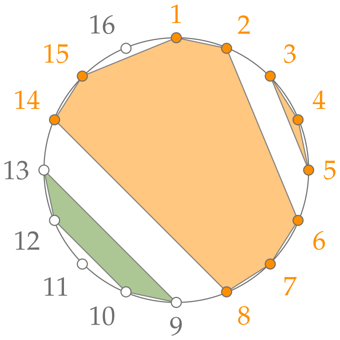

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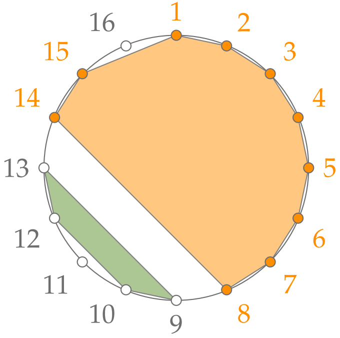

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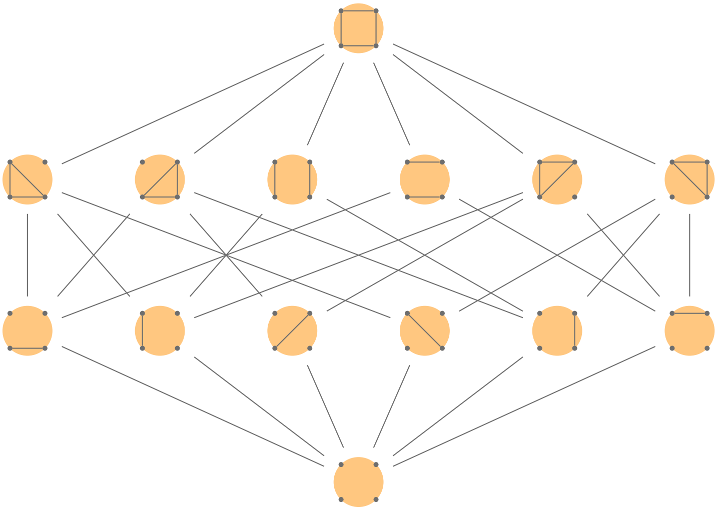
Theorem (G. Kreweras, 1972)

For $n \geq 0$, the poset $(NC_n, \leq_{\mathrm{dref}})$ is a lattice.

Example: $(NC_4, \leq_{\text{dref}})$



A Bijection

- if $\mathbf{x} \lessdot_{\text{dref}} \mathbf{y}$, then there are $B, B' \in \mathbf{x}$ such that

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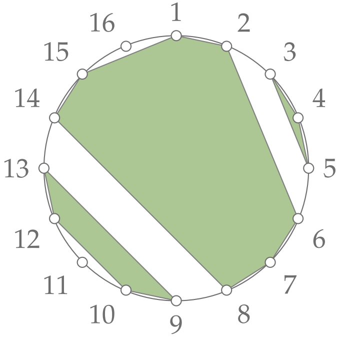

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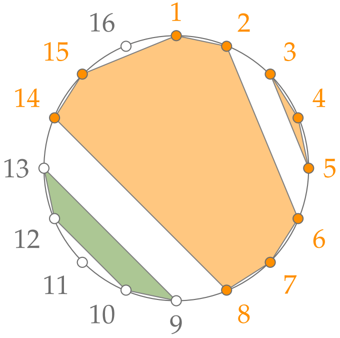

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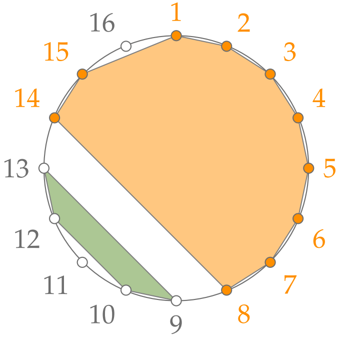

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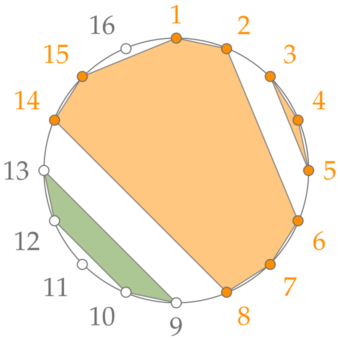

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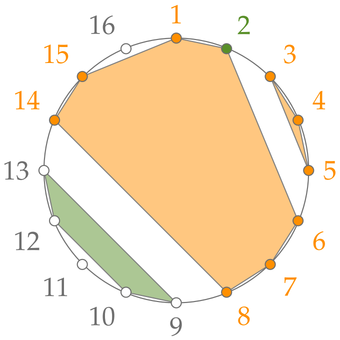

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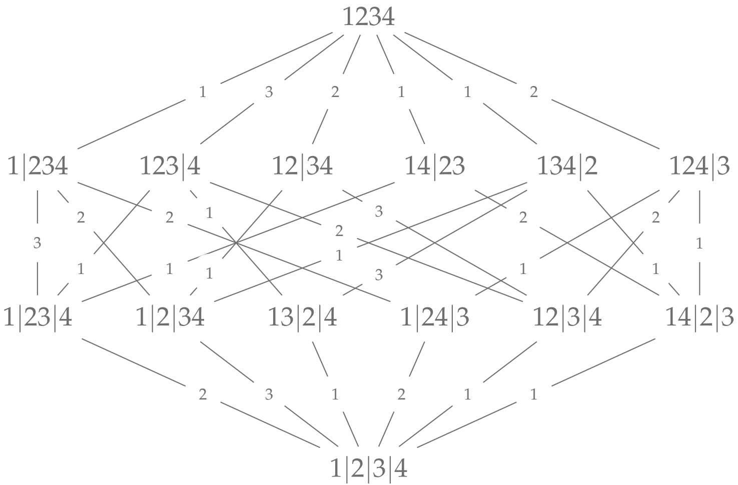
Theorem (R. Stanley, 1997; P. Biane, 2001)

The map π is a bijection from $\mathscr{C}_n$ to $\mathbb{PF}_{n-1}$.

Example: $(NC_4, \leq_{\mathrm{dref}})$



Outline

What about $\mathbb{PF}_{n,k}$?

Idea (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $1 < k < n$ study the subposet of $(NC_n, \leq_{\text{dref}})$ induced by the maximal chains in $\mathbb{PF}_{n-1,k}$.

- denote this poset by $\mathcal{P}_{n,k}$

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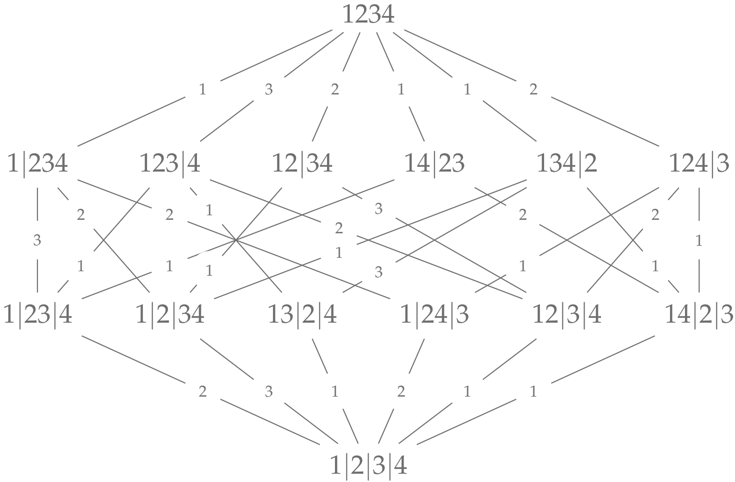
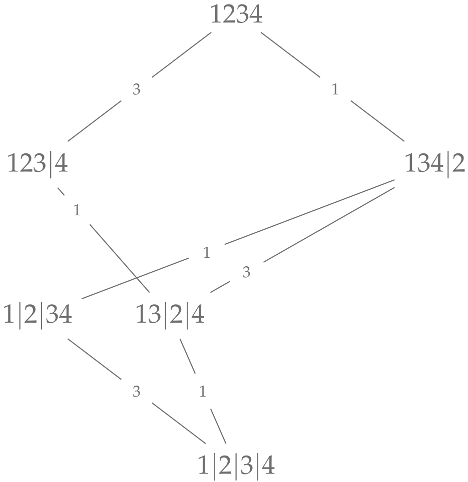


Example: $\mathcal{P}_{4,2}$



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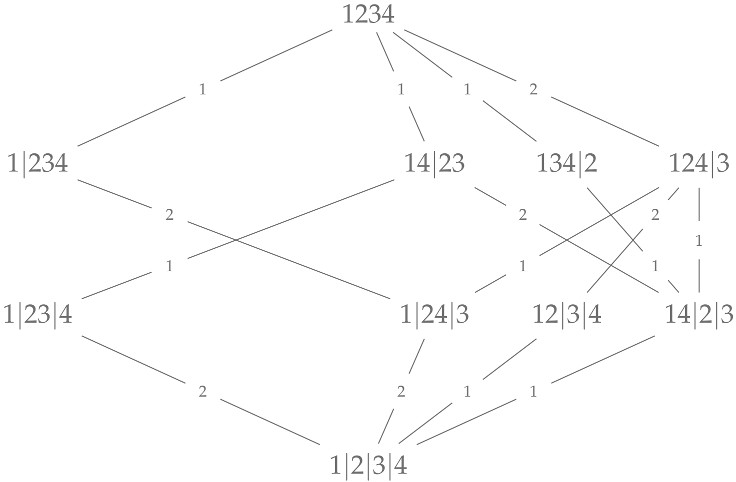

Some Properties

- $\mathcal{B}_n$ .. Boolean lattice of rank n

Theorem (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

If $n > k$, then $\mathcal{P}_{n,k} \cong \mathcal{P}_{k+1,k} \times \mathcal{B}_{n-k-1}$.

Some Properties

- **Möbius function**:

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & x = y \\ -\sum_{x \leq z < y} \mu(x,z), & x < y \\ 0, & \text{otherwise} \end{cases}$$

- let $\mathbf{0} = 1|2|\cdots|n$ and $\mathbf{1} = 123\cdots n$

Theorem (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $1 < k < n$ we have $\mu_{\mathcal{P}_{n,k}}(\mathbf{0},\mathbf{1}) = 0$.

Some Properties

- **order complex**: simplicial complex whose faces are chains

Conjecture (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

The order complex of $\mathcal{P}_{n,k} \setminus \{\mathbf{0}, \mathbf{1}\}$ *is contractible.*

Outline

The Elements of $\mathcal{P}_{n,n-1}$

- the Structure Theorem implies that it suffices to study $\mathcal{P}_{n,n-1}$

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- define $$X_n = \big\{\mathbf{x} \in NC_n \mid \{n-1, n\} \in \mathbf{x}\big\}$$ $$Y_n = \big\{\mathbf{x} \in NC_n \mid \{n\} \in \mathbf{x} \text{ and } 1 \sim_{\mathbf{x}} n-1\big\}$$
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Lemma (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $n \geq 3$ the ground set of $\mathcal{P}_{n,n-1}$ is precisely PE_n.

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Corollary

We have $|PE_3| = 3$ *and for* $n \geq 4$

$$|PE_n| = Cat(n) - 2Cat(n-2).$$

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Corollary

We have $|PE_3| = 3$ *and for* $n \geq 4$

$$|PE_n| = \left(\frac{5}{n+1} + \frac{9}{n-3}\right)\binom{2n-4}{n-4}.$$

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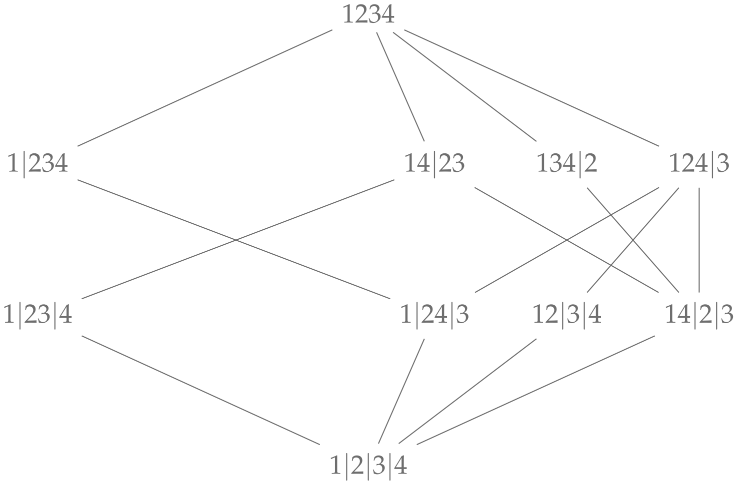
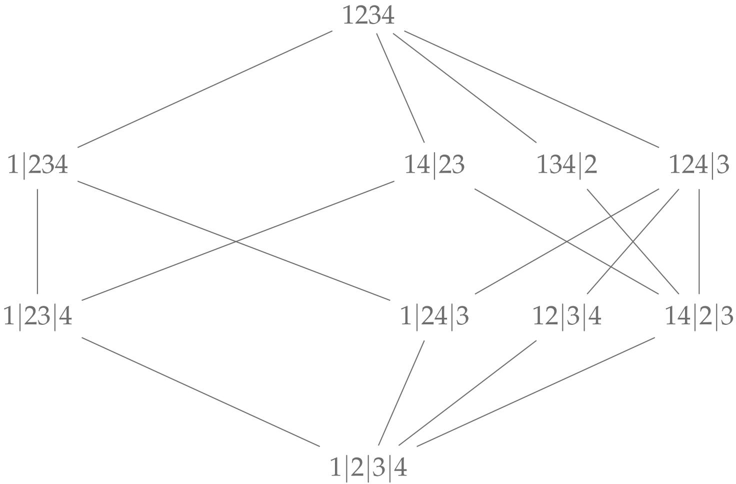
- How about we study the poset $(PE_n, \leq_{\text{dref}})$ a bit?

Example: $\mathcal{P}_{4,3}$



Example: $(PE_4, \leq_{\mathrm{dref}})$



Some Properties

Theorem (, 2017)

For $n \geq 3$ the poset $(\mathcal{PE}_n, \leq_{\mathrm{dref}})$ is a graded lattice.

Some Properties

- **left-modular**: $\mathbf{x}$ that satisfies $(\mathbf{y} \vee \mathbf{x}) \wedge \mathbf{z} = \mathbf{y} \vee (\mathbf{x} \wedge \mathbf{z})$ for all $\mathbf{y} \leq \mathbf{z}$

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Proposition (, 2017)

For $i \in [n]$ the element $\mathbf{x}_i$ is left-modular in $(\mathcal{PE}_n, \leq_{\text{dref}})$.

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Corollary

For $n \geq 3$ the lattice $(\mathcal{PE}_n, \leq_{\text{dref}})$ is supersolvable.

Some Properties

- for $\mathbf{y} \lessdot_{\mathrm{dref}} \mathbf{z}$ define

$$\lambda(\mathbf{y},\mathbf{z}) = \min\{i \mid \mathbf{z} = \mathbf{y} \vee \mathbf{x}_i \wedge \mathbf{z}\} - 1$$

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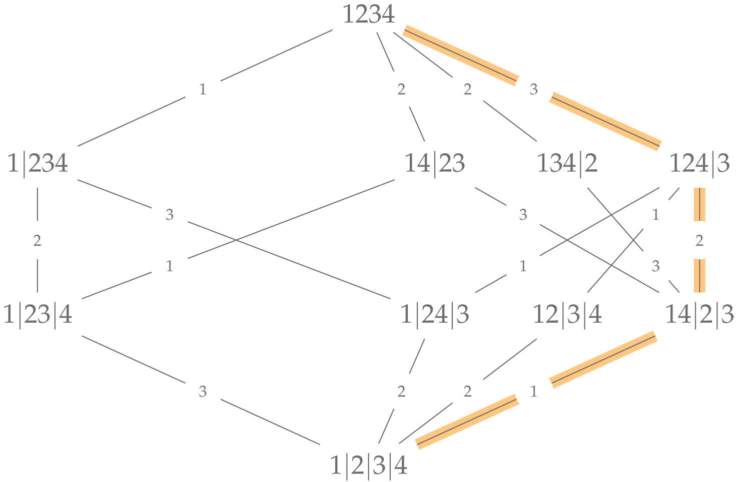
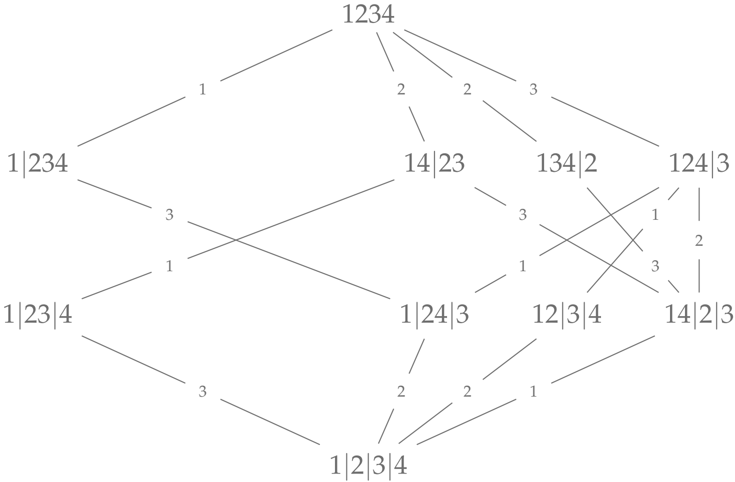
Corollary

For $n \geq 3$ the map λ is an EL-labeling of $(PE_n, \leq_{\text{dref}})$.

Example: $(PE_4, \leq_{\mathrm{dref}})$



Example: $\mathcal{P}_{4,3}$



Solving the Conjecture

Proposition (⚙, 2017)

For $n \geq 3$, the map λ restricts to an EL-labeling of $\mathcal{P}_{n,n-1}$.

Solving the Conjecture

- recall: $\mathcal{P}_{n,k} \cong \mathcal{P}_{k+1,k} \times \mathcal{B}_{n-k-1}$

Corollary

For $1 < k < n$ there exists an EL-labeling for $\mathcal{P}_{n,k}$.

Solving the Conjecture

- recall: $\mu_{\mathcal{P}_{n,k}}(\mathbf{0},\mathbf{1}) = 0$

Corollary

For $1 < k < n$ the order complex of $\mathcal{P}_{n,k} \setminus \{\mathbf{0},\mathbf{1}\}$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres. The number of these spheres is given by $\mu_{\mathcal{P}_{n,k}}(\mathbf{0},\mathbf{1})$.

Solving the Conjecture

- recall: $\mu_{\mathcal{P}_{n,k}}(\mathbf{0},\mathbf{1}) = 0$

Corollary

For $1 < k < n$ the order complex of $\mathcal{P}_{n,k} \setminus \{\mathbf{0},\mathbf{1}\}$ is contractible.

Thank You.

Outline

What about $\mu_{(PE_n,\leq_{\text{dref}})}(\mathbf{0},\mathbf{1})$?

- $\mathbf{a}_{i,j}$ .. noncrossing partition with only non-singleton block $\{i,j\}$
- $\bar{A}_n = \{\mathbf{a}_{i,j} \mid 1 \leq i < j \leq n\} \setminus \{\mathbf{a}_{1,n-1}, \mathbf{a}_{n-1,n}\}$
- let $\trianglelefteq$ be any partial order on $\bar{A}_n$; $X \subseteq \bar{A}_n$

What about $\mu_{(PE_n,\leq_{\text{dref}})}(\mathbf{0},\mathbf{1})$?

- **bounded below**: for every $\mathbf{x} \in X$ there is $\mathbf{a} \in \bar{\mathcal{A}}_n$ such that $\mathbf{a} \lessdot \mathbf{x}$ and $\mathbf{a} <_{\text{dref}} \bigvee X$

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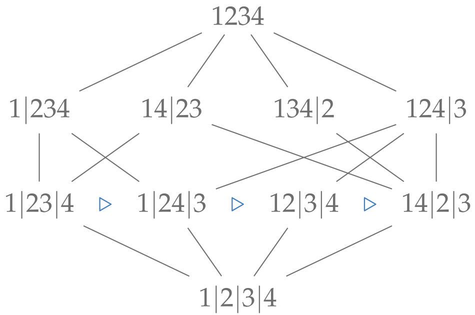

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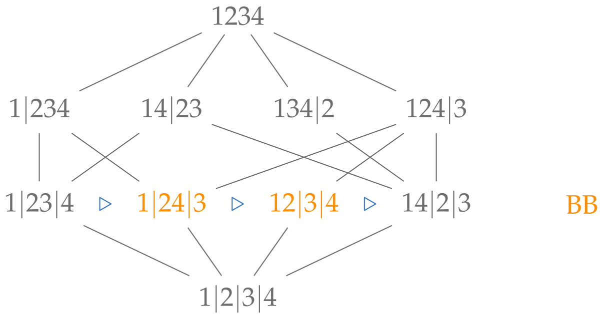

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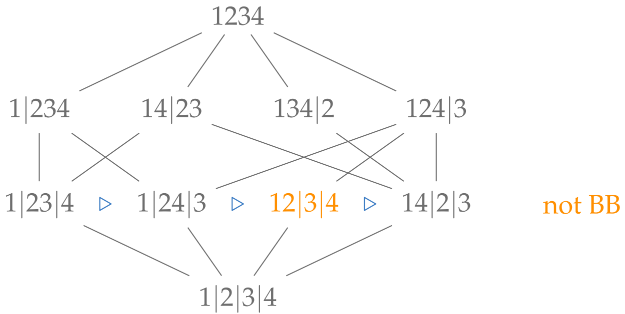

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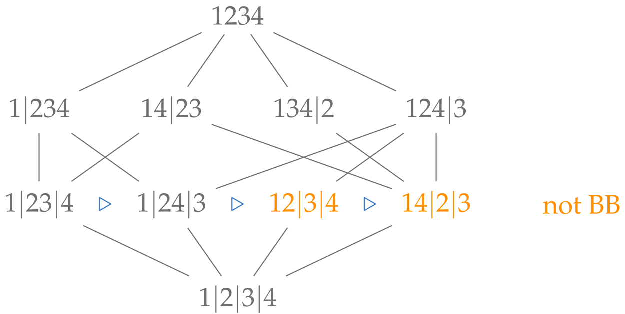

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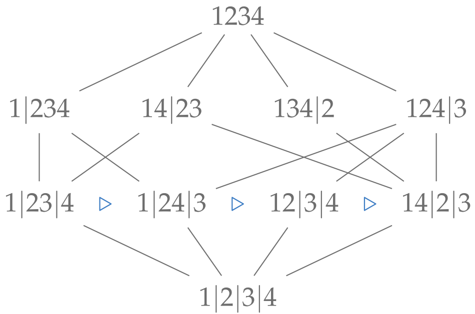
- **NBB**: no nonempty subset of X is BB
- **NBB-base** for $\mathbf{x}$: X is NBB and $\bigvee X = \mathbf{x}$



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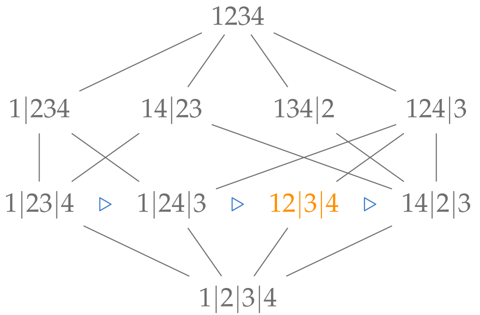

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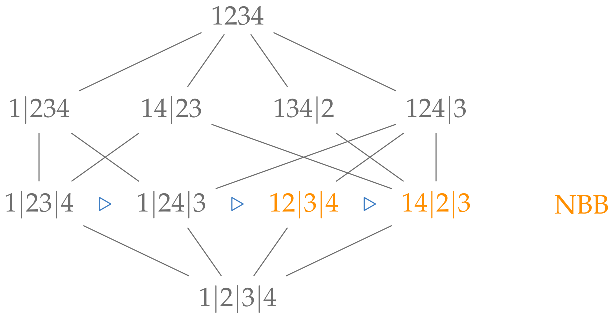

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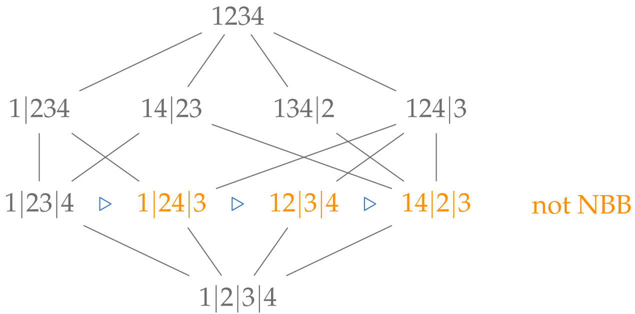

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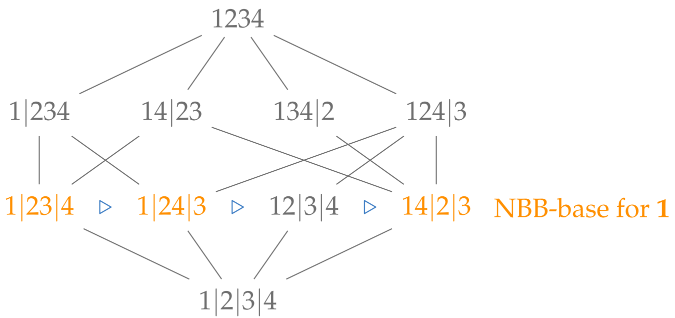

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Theorem (A. Blass, B. Sagan, 1997)

Let $\mathcal{P} = (P, \leq)$ be a finite lattice and $\trianglelefteq$ any partial order on the atoms of $\mathcal{P}$. For $x \in P$ we have

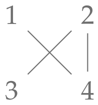
$$\mu_{\mathcal{P}}(\hat{0}, x) = \sum_X (-1)^{|X|},$$

where the sum runs over the NBB-bases for x.

NBB-Bases for $\mathbf{1}$ in $(PE_n, \leq_{\text{dref}})$

- subsets of $\bar{A}_n$ correspond to certain graphs on $[n]$

$$\{\mathbf{a}_{1,4}, \mathbf{a}_{2,3}, \mathbf{a}_{2,4}\} \quad \leftrightarrow$$



NBB-Bases for $\mathbf{1}$ in $(PE_n, \leq_{\text{dref}})$

- let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be the left-modular chain from before
- let $A_i = \{\mathbf{a} \in \bar{\mathcal{A}}_n \mid \mathbf{a} \not\leq_{\text{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \leq_{\text{dref}} \mathbf{x}_{i+1}\}$

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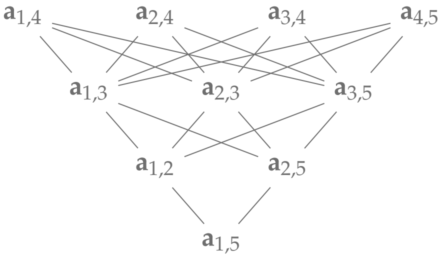

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Proposition (, 2017)

For $n \geq 3$ the NBB-bases for $\mathbf{1}$ in $(\mathcal{PE}_n, \leq_{\text{dref}})$ are precisely those maximal chains of $(\bar{\mathcal{A}}_n, \trianglelefteq)$, whose associated graph is a tree with an edge between 1 and n such that:

- *the removal of this edge yields two trees on vertices $[k]$ and $\{k+1, k+2, \ldots, n\}$ for some $k \in [n-2]$, and*
- *there is no edge between $n-1$ and n.*

NBB-Bases for $\mathbf{1}$ in $(PE_n, \leq_{\mathrm{dref}})$

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Corollary

For $n \geq 3$ we have

$$\mu_{(PE_n, \leq_{\mathrm{dref}})}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1}\Big(Cat(n-1) - 2Cat(n-2)\Big).$$

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Corollary

For $n \geq 3$ we have

$$\mu_{(PE_n, \leq_{\text{dref}})}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1} \frac{4}{n} \binom{2n-5}{n-4}.$$

Type B

- **parking function of type B**: a map $f : [n] \to [n]$ $\rightsquigarrow \mathbb{PF}_n^B$
- **noncrossing partition of type B**: noncrossing partition of $[2n]$ symmetric under rotation by 180° $\rightsquigarrow NC_n^B$
- $\mathscr{C}_n^B$ .. maximal chains of $(NC_n^B, \leq_{\text{dref}})$

Theorem (P. Biane, 2001)

There is a bijection from $\mathscr{C}_n^B$ to $\mathbb{PF}_n^B$.

Type B

- **k-avoiding parking function of type B**: $f \in \mathbb{PF}_n^B$ with $k \notin f$, but $l \in f$ for all $l > k$
- $\mathcal{P}_{n,k}^B$ .. poset induced by $\mathbb{PF}_{n,k}^B$
- PE_n^B .. ground set of $\mathcal{P}_{n,n}^B$

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Conjecture (, 2017)

For $n \geq 0$, we have $\mu_{\mathcal{P}_{n,n}^B}(\mathbf{0}, \mathbf{1}) = 0$.

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Conjecture (, 2017)

For $n \geq 0$, we have
$$\left|PE_n^B\right| = \binom{2n}{n} - 3\binom{2n-3}{n-1}.$$

Type B

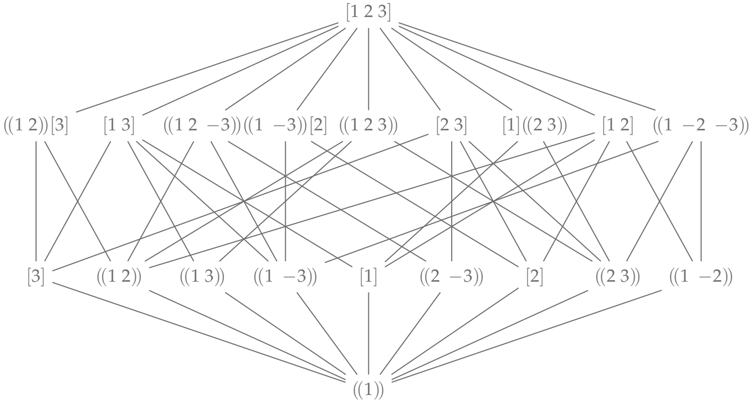
- **k-avoiding parking function of type B**: $f \in \mathbb{PF}_n^B$ with $k \notin f$, but $l \in f$ for all $l > k$
- $\mathcal{P}_{n,k}^B$ .. poset induced by $\mathbb{PF}_{n,k}^B$
- PE_n^B .. ground set of $\mathcal{P}_{n,n}^B$

Conjecture (, 2017)

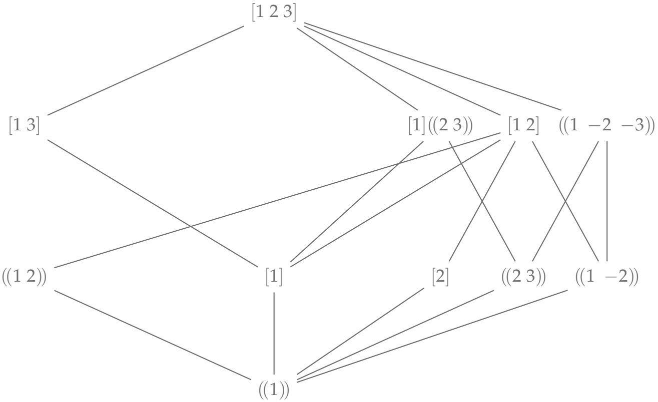
For $n \geq 0$, we have

$$\mu_{(PE_n^B, \leq_{\text{dref}})}(\mathbf{0}, \mathbf{1}) = (-1)^n \binom{2n-3}{n-3}.$$

Example: $\left(NC_3^B, \leq_{\text{dref}}\right)$



Example: $\left(PE_3^B, \leq_{\text{dref}}\right)$



Example: $\mathcal{P}^B_{3,3}$