Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

Henri Mühle

Basics
Parking Functions
Noncrossing Partitions
A Subposet of Noncrossing Partitions
Another Subposet of Noncrossing Partitions

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Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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Outline

1. Basics
   - Parking Functions
   - Noncrossing Partitions

2. A Subposet of Noncrossing Partitions

3. Another Subposet of Noncrossing Partitions
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3. Another Subposet of Noncrossing Partitions
Parking Functions

- $[n] = \{1, 2, \ldots, n\}$
- **parking function**: a map $f : [n] \to [n]$ such that for all $k \in [n]$ the set $f^{-1}([k])$ has at least $k$ elements
- $\mathbb{PF}_n$ .. set of all parking functions of length $n$
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- \(\mathbb{PF}_n\) .. set of all parking functions of length \(n\)

- \(\mathbb{PF}_3:\)

  \[
  \begin{align*}
  (1, 1, 1) & \quad (1, 1, 2) & \quad (1, 1, 3) & \quad (1, 2, 1) & \quad (1, 2, 2) & \quad (1, 2, 3) \\
  (1, 3, 1) & \quad (1, 3, 2) & \quad (2, 1, 1) & \quad (2, 1, 2) & \quad (2, 1, 3) & \quad (2, 2, 1) \\
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Parking Functions

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- **parking function**: a map \(f: [n] \to [n]\) such that for all \(k \in [n]\) the set \(f^{-1}([k])\) has at least \(k\) elements
- \(\mathbb{PF}_n\) : set of all parking functions of length \(n\)

**Theorem (Folklore)**

For \(n \geq 0\), the cardinality of \(\mathbb{PF}_n\) is \((n + 1)^{n-1}\).
Parking Functions

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Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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Parking Functions

Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions
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- **$k$-avoiding parking function**: $f \in \text{PF}_n$ with $k \notin f$, but $l \in f$ for all $l > k$
- $\text{PF}_{n,k}$ .. set of all $k$-avoiding parking functions
Undesired Parking Spaces

- **\(k\)-avoiding parking function**: \(f \in \mathbb{PF}_n\) with \(k \notin f\), but \(l \in f\) for all \(l > k\)
- \(\mathbb{PF}_{n,k}\) : set of all \(k\)-avoiding parking functions

- \(\mathbb{PF}_3:\)
  
  \[
  (1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,1) \quad (1,2,2) \quad (1,2,3) \\
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  (2,3,1) \quad (3,1,1) \quad (3,1,2) \quad (3,2,1)
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Undesired Parking Spaces

- **$k$-avoiding parking function**: $f \in \mathbb{PF}_n$ with $k \not\in f$, but $l \in f$ for all $l > k$

- $\mathbb{PF}_{n,k}$ .. set of all $k$-avoiding parking functions

\[
\mathbb{PF}_{3,1}:
\begin{align*}
(1,1,1) & \quad (1,1,2) & \quad (1,1,3) & \quad (1,2,1) & \quad (1,2,2) & \quad (1,2,3) \\
(1,3,1) & \quad (1,3,2) & \quad (2,1,1) & \quad (2,1,2) & \quad (2,1,3) & \quad (2,2,1) \\
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\end{align*}
\]
Undesired Parking Spaces

- **k-avoiding parking function**: $f \in \text{PF}_n$ with $k \notin f$, but $l \in f$ for all $l > k$

- $\text{PF}_{n,k}$: set of all $k$-avoiding parking functions

- $\text{PF}_{3,2}$:
  
  $$(1,1,1) \quad (1,1,2) \quad (1,1,3) \quad (1,2,1) \quad (1,2,2) \quad (1,2,3)$$
  $$(1,3,1) \quad (1,3,2) \quad (2,1,1) \quad (2,1,2) \quad (2,1,3) \quad (2,2,1)$$
  $$(2,3,1) \quad (3,1,1) \quad (3,1,2) \quad (3,2,1)$$
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- **k-avoiding parking function**: $f \in \mathcal{PF}_n$ with $k \notin f$, but $l \in f$ for all $l > k$

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- $\mathcal{PF}_{3,3}$:

  - $(1, 1, 1)$  
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**Undesired Parking Spaces**

- **\( k \)-avoiding parking function**: \( f \in \mathcal{PF}_n \) with \( k \notin f \), but \( l \in f \) for all \( l > k \)
- \( \mathcal{PF}_{n,k} \): set of all \( k \)-avoiding parking functions

**Proposition (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)**

For \( n \geq 0 \) and \( k \in [n] \), the cardinality of \( \mathcal{PF}_{n,k} \) is

\[
\frac{n!}{k!} | \mathcal{PF}_{k,k} |.
\]
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**Proposition (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)**

For \( n \geq 0 \) and \( k \in [n] \), the cardinality of \( \mathbb{PF}_{n,k} \) is

\[
\frac{n!}{k!} \left( (k + 1)^{k-1} - k^{k-1} \right).
\]
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2 A Subposet of Noncrossing Partitions

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Basics
Parking Functions
Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

Noncrossing Partitions ⇒ $\sim \sim N C_n$
Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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Basics
Parking Functions

Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

- **noncrossing partition** $\mapsto \mathcal{N}_n$

\[\{\{1, 6, 7\}, \{2, 8, 14, 15\}, \{3, 4, 5\}, \{9, 10, 12, 13\}, \{11\}, \{16\}\}\]
Noncrossing Partitions

- noncrossing partition

\[ \{ \{1, 6, 7\}, \{2, 8, 14, 15\}, \{3, 4, 5\}, \{9, 10, 12, 13\}, \{11\}, \{16\} \} \]
Noncrossing Partitions

Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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Basics
Parking Functions
Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

noncrossing partition

\[ \sim \rightarrow \mathcal{NC}_n \]

\( \{1, 6, 7\}, \{2, 8, 14, 15\}, \{3, 4, 5\}, \{9, 10, 12, 13\}, \{11\}, \{16\} \)
Noncrossing Partitions

\[ \{1, 2, 6, 7, 8, 14, 15\}, \{3, 4, 5\}, \{9, 10, 12, 13\}, \{11\}, \{16\} \]
Noncrossing Partitions

Theorem (G. Kreweras, 1972)

For $n \geq 0$, the cardinality of $\text{NC}_n$ is

$$\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}.$$
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Basics
Parking Functions
Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

- **dual refinement order**

\[ \sim \leq_{dref} \]

\[ \{\{1,2,6,7,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\} \]
Noncrossing Partitions

- dual refinement order

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Basics
Parking Functions
Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

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dual refinement order

\[ \rightsquigarrow \leq_{dref} \]

\( \{1, 2, 3, 4, 5, 6, 7, 8, 14, 15\} , \{9, 10, 12, 13\} , \{11\} , \{16\} \)
Theorem (G. Kreweras, 1972)

For \( n \geq 0 \), the poset \((\mathcal{NC}_n, \leq_{\text{dref}})\) is a lattice.
Example: \((\mathcal{NC}_4, \leq_{\text{dref}})\)
A Bijection

- if $x \lessdot_{dref} y$, then there are $B, B' \in x$ such that

$$y = (x \setminus \{B, B'\}) \cup (B \cup B')$$
A Bijection

- if $x \preceq_{dref} y$, then there are $B, B' \in x$ such that

$$y = (x \setminus \{B, B'\}) \cup (B \cup B')$$
A Bijection

- If $x \preceq_{\text{dref}} y$, then there are $B, B' \in x$ such that

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A Bijection

if \( x \preceq_{\text{dref}} y \), then there are \( B, B' \in x \) such that

\[
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\]
if $x \preceq \text{dref } y$, then there are $B, B' \in x$ such that

$$y = (x \setminus \{B, B'\}) \cup (B \cup B')$$

define $\pi(x, y) = \max\{i \in B \mid i \leq j \text{ for all } j \in B'\}$

(wlog $\min B < \min B'$)
A Bijection

- if $x \preceq_{dref} y$, then there are $B, B' \in x$ such that
  \[ y = (x \setminus \{B, B'\}) \cup (B \cup B') \]

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A Bijection

- if \( x \prec_{\text{dref}} y \), then there are \( B, B' \in x \) such that
  \[
y = (x \setminus \{B, B'\}) \cup (B \cup B')
  \]
- define \( \pi(x, y) = \max\{i \in B \mid i \leq j \text{ for all } j \in B'\} \)
  \(\text{ (wlog } \min B < \min B')\)
A Bijection

- $\pi$ extends to a labeling of the maximal chains of $(NC_n, \leq_{\text{dref}}) \mapsto C_n$
A Bijection

- \(\pi\) extends to a labeling of the maximal chains of \((\mathcal{NC}_n, \leq_{\text{dref}})\)

Theorem (R. Stanley, 1997; P. Biane, 2001)

The map \(\pi\) is a bijection from \(\mathcal{C}_n\) to \(\mathcal{PF}_{n-1}\).
Example: \((\mathcal{NC}_4, \leq_{d_{\text{ref}}})\)
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What about $\mathcal{PF}_{n,k}$?

Idea (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $1 < k < n$ study the subposet of $(\mathcal{NC}_n, \leq_{dref})$ induced by the maximal chains in $\mathcal{PF}_{n-1,k}$.

• denote this poset by $\mathcal{P}_{n,k}$
Example: \((\mathcal{NC}_4, \leq_{\text{dref}})\)
Example: $P_{4,2}$
Example: $\mathcal{P}_{4,3}$
Some Properties

- $B_n$ .. Boolean lattice of rank $n$

Theorem (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

If $n > k$, then $\mathcal{P}_{n,k} \cong \mathcal{P}_{k+1,k} \times B_{n-k-1}$. 
Some Properties

- **Möbius function:**

\[
\mu_P(x, y) = \begin{cases} 
1, & x = y \\
- \sum_{x \leq z < y} \mu(x, z), & x < y \\
0, & \text{otherwise}
\end{cases}
\]

- let \(0 = 1|2|\cdots|n\) and \(1 = 123\cdots n\)

**Theorem (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)**

For \(1 < k < n\) we have \(\mu_{P_{n,k}}(0, 1) = 0\).
Some Properties

- **order complex**: simplicial complex whose faces are chains

Conjecture (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

*The order complex of $\mathcal{P}_{n,k} \setminus \{0,1\}$ is contractible.*
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The Elements of $\mathcal{P}_{n,n-1}$

- the Structure Theorem implies that it suffices to study $\mathcal{P}_{n,n-1}$
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write $i \sim_x j$ if there exists $B \in x$ with $i, j \in B$

define $X_n = \{ x \in NC_n \mid \{n-1, n\} \in x \}$

$Y_n = \{ x \in NC_n \mid \{n\} \in x$ and $1 \sim_x n-1 \}$

let $PE_n = NC_n \setminus (X_n \cup Y_n)$
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Parking Functions Noncrossing Partitions

A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

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- define $X_n = \{ x \in \text{NC}_n \mid \{n-1, n\} \in x \}$
  
  $Y_n = \{ x \in \text{NC}_n \mid \{n\} \in x \text{ and } 1 \sim_x n-1 \}$
- let $\mathcal{P}E_n = \text{NC}_n \setminus (X_n \cup Y_n)$

Lemma (M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo & I. Nicolas, 2016)

For $n \geq 3$ the ground set of $\mathcal{P}_{n,n-1}$ is precisely $\mathcal{P}E_n$. 
The Elements of $\mathcal{P}_{n,n-1}$

- the Structure Theorem implies that it suffices to study $\mathcal{P}_{n,n-1}$
- write $i \sim_x j$ if there exists $B \in x$ with $i, j \in B$
- define $X_n = \{x \in \mathcal{NC}_n \mid \{n-1, n\} \in x\}$
  
  $Y_n = \{x \in \mathcal{NC}_n \mid \{n\} \in x \text{ and } 1 \sim_x n-1\}$
- let $PE_n = \mathcal{NC}_n \setminus (X_n \cup Y_n)$

Corollary

We have $|PE_3| = 3$ and for $n \geq 4$

$$|PE_n| = \text{Cat}(n) - 2\text{Cat}(n-2).$$
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The Elements of $\mathcal{P}_{n,n-1}$

- the Structure Theorem implies that it suffices to study $\mathcal{P}_{n,n-1}$
- write $i \sim_{x} j$ if there exists $B \in x$ with $i, j \in B$
- define $X_n = \{ x \in \text{NC}_n | \{ n-1, n \} \in x \}$
  $Y_n = \{ x \in \text{NC}_n | \{ n \} \in x$ and $1 \sim_{x} n - 1 \}$
- let $PE_n = \text{NC}_n \setminus (X_n \cup Y_n)$

Corollary

We have $|PE_3| = 3$ and for $n \geq 4$

$$|PE_n| = \left( \frac{5}{n+1} + \frac{9}{n-3} \right) \left( \frac{2n-4}{n-4} \right).$$
The Elements of $\mathcal{P}_{n,n-1}$

- the Structure Theorem implies that it suffices to study $\mathcal{P}_{n,n-1}$

- write $i \sim_x j$ if there exists $B \in x$ with $i, j \in B$

- define $X_n = \{ x \in \text{NC}_n | \{ n-1, n \} \in x \}$

- $Y_n = \{ x \in \text{NC}_n | \{ n \} \in x$ and $1 \sim_x n-1 \}$

- let $\text{PE}_n = \text{NC}_n \setminus (X_n \cup Y_n)$

- How about we study the poset $(\text{PE}_n, \leq_{dref})$ a bit?
Example: \( \mathcal{P}_{4,3} \)
Example: \((PE_4, \leq_{dref})\)
Theorem (Mühle, 2017)

For $n \geq 3$ the poset $(\mathcal{PE}_n, \leq_{dref})$ is a graded lattice.
Some Properties

- **left-modular**: \( x \) that satisfies \((y \lor x) \land z = y \lor (x \land z)\) for all \( y \leq z \)
Some Properties

- **left-modular**: $x$ that satisfies $(y \lor x) \land z = y \lor (x \land z)$ for all $y \leq z$

- $x_i$ .. noncrossing partition with only non-singleton block $[i - 1] \cup \{n\}$
Some Properties

- **left-modular**: $x$ that satisfies $(y \lor x) \land z = y \lor (x \land z)$ for all $y \leq z$
- $x_i$ .. noncrossing partition with only non-singleton block $[i - 1] \cup \{n\}$

**Proposition (Mühle, 2017)**

For $i \in [n]$ the element $x_i$ is left-modular in $(PE_n, \leq_{dref})$. 
Some Properties

- **left-modular**: $x$ that satisfies $(y \lor x) \land z = y \lor (x \land z)$ for all $y \leq z$
- $x_i$ is a noncrossing partition with only non-singleton block $[i - 1] \cup \{n\}$

**Corollary**

*For $n \geq 3$ the lattice $(\mathcal{P}E_n, \leq_{\text{dref}})$ is supersolvable.*
Some Properties

- for $y \preceq_{dref} z$ define

$$\lambda(y, z) = \min\{i \mid z = y \lor x_i \land z\} - 1$$
Some Properties

- for \( y \triangleleft_{d\text{ref}} z \) define

\[
\lambda(y, z) = \min\{ i \mid z = y \lor x_i \land z \} - 1
\]

Corollary

For \( n \geq 3 \) the map \( \lambda \) is an EL-labeling of \((PE_n, \leq_{d\text{ref}})\).
Example: \((PE_4, \leq_{\text{dref}})\)
Example: $P_{4,3}$
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A Subposet of Noncrossing Partitions

Another Subposet of Noncrossing Partitions

Proposition (‡, 2017)

For $n \geq 3$, the map $\lambda$ restricts to an EL-labeling of $\mathcal{P}_{n,n-1}$. 
Solving the Conjecture

- recall: $\mathcal{P}_{n,k} \cong \mathcal{P}_{k+1,k} \times \mathcal{B}_{n-k-1}$

Corollary

For $1 < k < n$ there exists an EL-labeling for $\mathcal{P}_{n,k}$. 
Solving the Conjecture

- recall: $\mu_{P_{n,k}}(0,1) = 0$

**Corollary**

For $1 < k < n$ the order complex of $P_{n,k} \setminus \{0,1\}$ is homotopy equivalent to a wedge of $(n - 2)$-dimensional spheres. The number of these spheres is given by $\mu_{P_{n,k}}(0,1)$. 
Solving the Conjecture

- recall: $\mu_{P_{n,k}}(0,1) = 0$

Corollary

For $1 < k < n$ the order complex of $P_{n,k} \setminus \{0,1\}$ is contractible.
Thank You.
Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

Henri Mühle

Mobius Function

4 Möbius Function

5 Type B
What about $\mu(PE_n, \leq_{dref})(0, 1)$?

- $a_{i,j}$.. noncrossing partition with only non-singleton block $\{i, j\}$
- $\tilde{A}_n = \{a_{i,j} \mid 1 \leq i < j \leq n\} \setminus \{a_{1,n-1}, a_{n-1,n}\}$
- let $\leq$ be any partial order on $\tilde{A}_n; X \subseteq \tilde{A}_n$
What about $\mu(\text{PE}_n, \leq_{\text{dref}})(0,1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \triangleleft x$ and $a <_{\text{dref}} \bigvee X$
What about $\mu(PE_n, \leq_{dref})(0, 1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \nless x$ and $a <_{dref} \bigvee X$
What about $\mu(P_{E_n}, \leq_{\text{dref}})(0, 1)$?

- **bounded below:** for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \triangleleft x$ and $a <_{\text{dref}} \bigvee X$
What about $\mu(P_{E_n, \leq_{dref}})(0, 1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \lhd x$ and $a <_{dref} \bigvee X$
What about \( \mu_{(P_{E_n}, \leq_{dref})}(0, 1) \)?

- **bounded below**: for every \( x \in X \) there is \( a \in \bar{A}_n \) such that \( a \prec x \) and \( a <_{dref} \bigvee X \)
What about $\mu_{PE_n,\leq_{dref}}(0,1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \triangleleft x$ and $a <_{dref} \bigvee X$
- **NBB**: no nonempty subset of $X$ is BB
- **NBB-base** for $x$: $X$ is NBB and $\bigvee X = x$

Diagram:
What about $\mu_{(PE_n, \leq \text{dref})}(0, 1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \prec x$ and $a <_{\text{dref}} \bigvee X$
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![Diagram of posets](diagram.png)
What about $\mu_{(PE_n, \leq_{dref})}(0, 1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \triangleleft x$ and $a <_{dref} \bigvee X$
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```
1|234
 ▽
1|23|4 ▽ 1|24|3 ▽ 12|3|4 ▽ 14|2|3
 ▼
1|2|3|4
```

not NBB
What about $\mu(PE_n, \leq_{dref})(0, 1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \triangleleft x$ and $a <_{dref} \bigvee X$
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- **NBB-base** for $x$: $X$ is NBB and $\bigvee X = x$

Diagram:

```
1234
1|234   14|23   134|2   124|3
   ▶      ▶      ▶      ▶
1|23|4   1|24|3   12|3|4   14|2|3  NBB-base for 1
1|2|3|4
```
What about $\mu(PE_n, \leq_{dref})(0, 1)$?

- **bounded below**: for every $x \in X$ there is $a \in \bar{A}_n$ such that $a \triangleleft x$ and $a <_{dref} \bigvee X$
- **NBB**: no nonempty subset of $X$ is BB
- **NBB-base** for $x$: $X$ is NBB and $\bigvee X = x$

**Theorem (A. Blass, B. Sagan, 1997)**

Let $\mathcal{P} = (P, \leq)$ be a finite lattice and $\leq$ any partial order on the atoms of $\mathcal{P}$. For $x \in P$ we have

$$\mu_{\mathcal{P}}(\hat{0}, x) = \sum_X (-1)^{|X|},$$

where the sum runs over the NBB-bases for $x$. 

Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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Möbius Function

Type $B$

NBB-Bases for 1 in $(PE_n, \leq_{dref})$

- subsets of $\bar{A}_n$ correspond to certain graphs on $[n]$

\[
\{a_{1,4}, a_{2,3}, a_{2,4}\} \iff \begin{array}{c}
1 \\
\times \\
3 \\
\end{array} \begin{array}{c}
2 \\
| \\
4 \\
\end{array}
\]
NBB-Bases for 1 in \((PE_n, \leq_{dref})\)

- Let \(\{x_1, x_2, \ldots, x_n\}\) be the left-modular chain from before.
- Let \(A_i = \{a \in \bar{A}_n \mid a \not\leq_{dref} x_i \text{ and } a \leq_{dref} x_{i+1}\}\).
NBB-Bases for $1$ in $(PE_n, \leq_{dref})$

- let $\{x_1, x_2, \ldots, x_n\}$ be the left-modular chain from before
- let $A_i = \{a \in \bar{A}_n \mid a \not\leq_{dref} x_i \text{ and } a \leq_{dref} x_{i+1}\}$
- let $a \preceq a'$ if and only if $a \in A_i, a' \in A_j \text{ and } i \leq j$
Let \( \{x_1, x_2, \ldots, x_n\} \) be the left-modular chain from before.

Let \( A_i = \{a \in \bar{A}_n \mid a \nleq_{\text{dref}} x_i \text{ and } a \leq_{\text{dref}} x_{i+1}\} \)

Let \( a \sqsubseteq a' \) if and only if \( a \in A_i, a' \in A_j \text{ and } i \leq j \)
NBB-Bases for 1 in \((PE_n, \leq_{\text{dref}})\)

- let \(\{x_1, x_2, \ldots, x_n\}\) be the left-modular chain from before
- let \(A_i = \{a \in \overline{A}_n \mid a \not\leq_{\text{dref}} x_i \text{ and } a \leq_{\text{dref}} x_{i+1}\}\)
- let \(a \sqsubseteq a'\) if and only if \(a \in A_i, a' \in A_j\) and \(i \leq j\)

Proposition (\(\bigcirc\), 2017)

For \(n \geq 3\) the NBB-bases for 1 in \((PE_n, \leq_{\text{dref}})\) are precisely those maximal chains of \((\overline{A}_n, \leq)\), whose associated graph is a tree with an edge between 1 and \(n\) such that:

- the removal of this edge yields two trees on vertices \([k]\) and \(\{k + 1, k + 2, \ldots, n\}\) for some \(k \in [n - 2]\), and
- there is no edge between \(n - 1\) and \(n\).
NBB-Bases for $1$ in $(PE_n, \leq_{dref})$

- let $\{x_1, x_2, \ldots, x_n\}$ be the left-modular chain from before
- let $A_i = \{a \in \bar{A}_n \mid a \not\leq_{dref} x_i \text{ and } a \leq_{dref} x_{i+1}\}$
- let $a \preceq a'$ if and only if $a \in A_i, a' \in A_j$ and $i \leq j$

**Corollary**

For $n \geq 3$ we have

$$\mu(PE_n, \leq_{dref})(0, 1) = (-1)^{n-1}\left(\text{Cat}(n - 1) - 2\text{Cat}(n - 2)\right).$$
NBB-Bases for 1 in \((PE_n, \leq_{\text{dref}})\)

- let \(\{x_1, x_2, \ldots, x_n\}\) be the left-modular chain from before
- let \(A_i = \{a \in \bar{A}_n \mid a \not\leq_{\text{dref}} x_i \text{ and } a \leq_{\text{dref}} x_{i+1}\}\)
- let \(a \preceq a'\) if and only if \(a \in A_i, a' \in A_j\) and \(i \leq j\)

Corollary

For \(n \geq 3\) we have

\[
\mu(PE_n, \leq_{\text{dref}})(0,1) = (-1)^{n-1} \frac{4}{n} \binom{2n-5}{n-4}.
\]
Type $B$

- **parking function of type $B$:** a map $f: [n] \rightarrow [n] \leadsto \mathbb{PF}_n$
- **noncrossing partition of type $B$:** noncrossing partition of $[2n]$ symmetric under rotation by $180^\circ \leadsto \mathbb{NC}_n$
- $\mathcal{C}_n^B$ .. maximal chains of $(\mathbb{NC}_n, \leq_{dref})$

**Theorem (P. Biane, 2001)**

There is a bijection from $\mathcal{C}_n^B$ to $\mathbb{PF}_n^B$. 
Type $B$

- **$k$-avoiding parking function of type $B$:** $f \in \mathcal{PF}_n^B$ with $k \notin f$, but $l \in f$ for all $l > k$

- $\mathcal{P}^B_{n,k}$ .. poset induced by $\mathcal{PF}^B_{n,k}$

- $\mathcal{PE}^B_n$ .. ground set of $\mathcal{P}^B_{n,n}$
Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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M"obius Function

Type B

- **k-avoiding parking function of type B**: $f \in \mathbb{PF}_n^B$ with $k \notin f$, but $l \in f$ for all $l > k$
- $P_n^B$ .. poset induced by $\mathbb{PF}_n^B$
- $PE_n^B$ .. ground set of $P_n^B$
Type B

- **k-avoiding parking function of type B**: \( f \in PF_B^n \) with \( k \not\in f \), but \( l \in f \) for all \( l > k \)
- \( P_{n,k}^B \) .. poset induced by \( PF_{n,k}^B \)
- \( PE_n^B \) .. ground set of \( P_{n,n}^B \)

**Conjecture (‡, 2017)**

For \( n \geq 0 \), we have \( \mu_{P_B^{n,n}}(0, 1) = 0 \).
Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

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Mobius Function

Type B

**Type B**

- **k-avoiding parking function of type B**: $f \in PF_n^B$ with $k \notin f$, but $l \in f$ for all $l > k$
- $P_{n,k}^B$.. poset induced by $PF_{n,k}^B$
- $PE_n^B$ .. ground set of $P_{n,n}^B$

**Conjecture (.sax, 2017)**

For $n \geq 0$, we have

$$|PE_n^B| = \binom{2n}{n} - 3 \binom{2n-3}{n-1}.$$
**Type B**

- **$k$-avoiding parking function of type B**: $f \in \mathcal{PF}_n^B$ with $k \not\in f$, but $l \in f$ for all $l > k$
- $\mathcal{P}_{n,k}^B$ .. poset induced by $\mathcal{PF}_{n,k}^B$
- $\mathcal{PE}_{n}^B$ .. ground set of $\mathcal{P}_{n,n}^B$

**Conjecture ( Harami, 2017)**

For $n \geq 0$, we have

$$\mu_{(\mathcal{PE}_n^B, \preceq_{\text{dref}})}(0, 1) = (-1)^n \binom{2n - 3}{n - 3}.$$
Example: \( (\mathcal{NC}_3^B, \leq_{dref}) \)
Example: \((PE^B_3, \leq_{\text{dref}})\)
Example: $\mathcal{P}_{3,3}^B$